Pricing Options on Realized Variance in Heston Model with Jumps in Returns and Volatility

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Abstract

We develop analytical methodology for pricing and hedging options on the realized variance under the Heston stochastic variance model (1993) augmented with jumps in asset returns and variance. By employing generalized Fourier transform we obtain analytical solutions (up to numerical inversion of Fourier integral) for swaps on the realized volatility and variance and for options on these swaps. We also extend our framework for pricing forward-start options on the realized variance and volatility, including options on the VIX. Our methodology allows us to consistently unify pricing and risk managing of different volatility options. We provide an example of model parameters estimation using both time series of the VIX and the VIX options data and find that the proposed model is in agreement with both historical and implied market data. Finally, we derive a log-normal approximation to the density of the realized variance in the Heston model and obtain accurate approximate solution for volatility options with longer maturities.

Keywords: realized variance, variance swap, volatility swap, option on variance swap, volatility derivatives, VIX futures, VIX option, Heston model.

1 Introduction

The market for options on the realized variance has been growing very rapidly in recent years. As a result, nowadays investors can trade in a variety of contracts starting from vanilla swaps on realized variance up to options on the future realized variance. Options on the realized volatility are attractive investment tools for investors with specific views on the future market volatility or with particular risk exposures by allowing them to deal with these views and risks without taking direct positions in the underlying asset and delta-hedging their positions.

The theoretical background for pricing options on the realized volatility has been developing since 1990th. In particular, Neuberger (1994) showed that by delta-hedging the log contract, which pays out the logarithm of the asset price at the contract maturity time, the investor accrues the difference between the realized variance and the implied variance used to calculate the delta-hedge. Whaley (1993) applies the Black (1976) formula for options on futures contracts to price options on an implied volatility index. Grunbichler-Longstaff (1996) employs the square root mean-reverting process to model the implied volatility index in continuous time setting and price options on it.

In an influential paper, Demeterfi *et al* (1999) provide a nice exposition of pricing and hedging variance swaps and, by summarizing results from previous studies, show how to replicate the variance swap through a dynamic trading strategy involving a portfolio of call and put options with appropriately chosen weights. This strategy allows a hedger to duplicate the pay-out of the variance swap in a model-independent fashion. Demeterfi *et al* also point out that for pricing and hedging more complex options on the realized variance with non-linear pay-off functions by means of dynamic trading in stock and other volatility options one needs to design an appropriate model describing the stochastic dynamics of the market volatility surface. One of the candidate models for this purpose is definitely the Heston stochastic volatility model (1993) which is nowadays an industry-wide model. To our knowledge, pricing volatility options under the Heston model with return and volatility jumps was first proposed by Matytsin (1999). Among others, Javaheri *et al* (2004), Howison *et al* (2004), and Elliott *et al* (2007) apply different stochastic volatility models for pricing and hedging of variance and volatility swaps.

As an alternative to stochastic volatility models, Carr *et al* (2005) consider pure jump processes with independent increments for pricing volatility options. Little-Pan (2001) value variance swaps in the local volatility framework (Dupire (1994)) by developing an appropriate finite-difference scheme. Windcliff *et al* (2006) apply a numerical approach to solve the partial integro-differential equation (PIDE) satisfied by the value function of the variance swap under the Merton log-normal jump-diffusion model (1976). Buehler (2006) develops a general framework for modeling the term structure of variance swaps. Albanese *et al* (2006) apply spectral methods for pricing volatility options in a non-parametric model.

In our study, we concentrate on the Heston model with jumps in returns and variance and obtain closed-form solutions for a wide range of options on the realized varince. Analytical solution through Fourier transform for the density of the realized variance in the Heston model was obtained by Lipton (2001). However, by employing his method we need to discretisize the state space of future realized variance and for each state evaluate its probability by inverting the Fourier integral (employing FFT leads to the total number of operations given by $O(N \ln N)$), and finally compute the expected value of the option by convoluting the pay-out in the given state with the probability of this state. We apply the generalized Fourier transform and reduce the computation of a single option price to numerical inversion of a single Fourier integral. The generalized Fourier transform was introduced to the field of financial engineering by Carr-Madan (1999), Lewis (2000), and Lipton (2001), and it proved to be a very powerful tool by solving pricing and calibration problems for equity options. Here we extend its use for pricing options on the realized variance.

The purpose of introducing jumps in returns and variance dynamics is to make the Heston model consistent with short-term variance swaps with cap protection for which market prices are typically lower than theoretical prices implied by the Heston model with no jumps. Among others, the empirical study of the VIX time series by Dotsis et al (2007) shows that jumps in the dynamics of the S&P 500 index variance are statistically significant. Variance jumps are also necessary to produce positive volatility skews implied from market prices of VIX options (see, for example, Sepp (2008)).

Let us also note that although our developed pricing method is a model-specific one as opposed to approaches proposed, for example, by Demeterfi *et al* (1999), Friz-Gatheral (2005), and Carr-Lee (2007), there are certain advantages of using it. First, since in practice there is only a limited number of market quotes to imply the volatility skew per a single maturity, we need to introduce an interpolation and extrapolation method to to get market implied volatilities at certain strikes, which is crucial for replicating strategy. As a result, we might ultimately end up introducing model-dependence in our replication methods. Second, since forward-start options on the realized variance, including options on the VIX, depend on the realized variance observed at two future times, it is hardly possible to construct reliable replicating portfolios for these options. Ultimately, a dynamic pricing model calibrated to be consistent with liquid prices of available derivative securities allows us to achieve better flexibility in modeling and analyzing model-implied distributions, prices, and hedges for volatility options.

In related studies, Broadie-Jain (2008) consider pricing and, especially, hedging of variance options under the Heston model by solving numerically the corresponding pricing equation. Independently, Fatone *et al* (2007) applied an approach similar to ours for pricing options on the realized variance using the Fourier transform technique. Also, Matytsin (1999) presented the Fourier transform approach for pricing volatility options under the Heston model with price and volatility jumps. Our analysis is more general than that of Matytsin (1999) and Fatone *et al* (2007) inasmuch as we study the impact of jumps in returns and volatility on the pricing of volatility options and extend our results to a wider range of volatility options.

To describe the stochastic evolution of asset return variation we choose the square-root diffusion, which is a central part of a few important financial models including the CIR interest rate model (Cox-Ingersoll-Ross (1985)), the Heston stochastic volatility model (Heston (1993)), and the general affine model (Duffie-Pan-Singleton (2000)). This attractiveness of the square-root diffusion is motivated by several essential properties including positivity, mean-reversion, and closed-form solution for the transition density. In particular, the availability of the closed-from solution for European option prices in the Heston model makes the calibration to market prices relatively quick and efficient. Combined with the ability to reproduce volatility smiles and skews, all this makes the Heston model a viable tool in many pricing applications, including equity and foreign exchange (Lipton (2002), Lewis (2000)), and interest rates (Andreasen (2006)).

This paper is organized as follows. In Section 2, we introduce notations, highlight some popular options on the realized variance, and present our pricing model. In Section 3, we consider general problem of pricing options on the realized volatility and variance and then derive analytical formulas for their values. In Section 4, we treat pricing forward-start options on the future realized variance. In section 5, we apply our method for pricing options on the VIX and consider a method to estimate model parameters using historical time series of the VIX. In Section 6, we develop a log-normal approximation for the density of cumulative variance in the Heston model and obtain approximate solutions for the pricing problem. In Section 7, we briefly discuss calculating risk parameters for volatility options. In Section 8, we present some numerical examples of applying our method for pricing some of volatility options considered in this paper.

2 Problem Formulation

2.1 Market Model

As a basis, we use the Heston (1993) stochastic volatility model to describe the joint evolution of the underlying asset price $S(t)$ and its variance $V(t)$ under the martingale (pricing) measure. While the stochastic variance model fits the longer-term behavior of the asset price, to adequately describe the short-term behavior of the asset price we augment the Heston model with return jumps (Bates (1996), Scott (1997)). In addition, Sepp (2008) shows that for the realistic modeling of the volatility skew observed in market prices of VIX options we also need to include jumps in the variance. As a result, our model is close in spirit to the model proposed by Duffie *at al* (2000), although for brevity we do not assume the correlation between the volatility and asset return jumps. It also important to note that if return and volatility jump times are assumed to be independent then the correlation between the asset price and its variance tends to decrease, which is not in line with empirical observations. Accordingly, we assume that jumps occur simultaneously in both returns and variance.

To sum up, the joint dynamics of the asset price $S(t)$ and its variance $V(t)$ are assumed to evolve under the risk-neutral (pricing) measure, $\mathbb{Q}$, as follows:

$$\left\{ \begin{array}{ll} dS(t) & = (r(t) - d(t) - \gamma m^j)S(t)dt + \sqrt{V(t)}S(t)dW^s(t) + \left(e^{J^s} - 1\right) S(t)dN(t), \ S(0) = S; \\ dV(t) & = \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t) + J^v dN(t), \ V(0) = V, \end{array} \right. \tag{1}$$

where $r(t)$ and $d(t)$ are deterministic risk-free interest and dividend yields, respectively, θ is a long-term variance, κ is a mean-reverting rate, ε is a volatility of variance, $W^s(t)$ and $W^v(t)$ are correlated Wiener processes with constant correlation parameter ρ, $N(t)$ is a Poisson process with constant intensity γ. The values of the state variables in the right side of (1) are assumed to be taken just before a jump in $N(t)$ has occurred.

We assume that amplitudes of return jump, J^s, and variance jump, J^v, have respectively normal and exponential distribution:

$$\varpi^s(J^s) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left\{-\frac{(J^s - \nu)^2}{2\delta^2}\right\}, \ \varpi^v(J^v) = \frac{1}{\eta} \exp\left\{-\frac{1}{\eta} J^v\right\}, \tag{2}$$

where ν is mean and δ is volatility of return jump J^s and η is mean of variance jump J^v.

Finally

$$m^j = \exp\left\{\nu + \frac{1}{2}\delta^2\right\} - 1 \tag{3}$$

is the compensator.

We note that Matytsin (1999) presented a similar model specification by assuming that the jump in volatility J^s is a constant. From a modeling point of view, the model with exponential jump size distribution is equivalent to the model with the constant jump. However, from a pure numerical point of view, the former model is easier to deal with.

In practice, the model is calibrated to a set of market prices of liquid European options for which theoretical model prices are computed by means of the Fourier inversion technique as proposed, for example, by Carr-Madan (1999), Duffie-Pan-Singleton (2000), Lewis (2000), Lipton (2002) and Kangro *at al* (2004).

2.2 Realized Variance

In our study we concentrate on pricing and hedging options on the annualized variance of $S(t)$ realized during time period from t_0 to t_N:

$$I_N(t_0, t_N) = \frac{AF}{N}\sum_{n=1}^{N}\left(\ln\frac{S(t_n)}{S(t_{n-1})}\right)^2, \tag{4}$$

where $S(t)$ is the asset closing price observed at annualized times t_0 (contract inception), .., t_N (maturity), N is the number of observations from contract inception up to expiration time, $\ln(S(t_n)/S(t_{n-1}))$ is a return realized over the period between t_{n-1} and t_n. AF is an annualization factor.

In practice, the annualization factor AF is specified in contract terms (typically, $AF = 252$), N is calculated based on the fixing schedule which starts at time t_0 and finishes at time t_N (typically, daily fixings). At valuation time t, $t \in [t_0, t_N]$, the annualized time-to-maturity, τ, is defined as $\tau = t_N - t$. In the following text, t_0, T, and τ stand for the annualized inception time, maturity time, and time-to-maturity, respectively.

Now we introduce the notion of the quadratic variation, which is closely connected to the definition of the realized variance (4). References for quadratic variation under jump-diffusion processes can be found, for example, in Cont-Tankov (2004). Here we state the main result: if the asset price is driven by (1), then the quadratic variation, or the realized variance of $S(t)$, realized during time period $[t_0, T]$, is denoted by $I(t_0, T)$ and given by:

$$I(t_0, T) = \lim_{n\to\infty}\sum_{t_i\in\pi^n}\left(\ln\frac{S(t_{i+1})}{S(t_i)}\right)^2 = \int_{t_0}^{T} V(t')dt' + \sum_{k=N(t_0)}^{N(T)}(J_k^s)^2, \tag{5}$$

where $\pi^n = \{t_0^n < t_1^n < .. < t_{n+1}^n\}$, with $t_0^n = t_0$ and $t_{n+1}^n = T$, is a sequence of partitions of interval $[t_0, T]$ such that $|\pi^n| = \sup_p |t_p^n - t_{p-1}^n| \to 0$ as $n \to \infty$, and convergence in (5) is in probability sense. Here, $(J_k^s)^2$ is the squared realization of k-th jump J_k^s that occurred at jump time t_k.

Accordingly, in continuous-time setting assuming that the asset price follows a diffusion process with stochastic variance (1), the discrete realized variance $I_N(t_0, T)$ can be approximated by continuous-time realized variance $I(t_0, T)$ as follows:

$$I_N(t_0, T) \to \hat{I}(t_0, T) = \frac{1}{T - t_0} I(t_0, T), \tag{6}$$

where $\hat{I}(t_0, T)$ is annualized realized variance accrued over period $[t_0, T]$, $I(t_0, T)$ is de-annualized realized variance. Here, we take into account the fact that the number of fixings is typically proportional to the annualization factor: $N \sim (T - t_0)AF$. For options on the future realized variance, the averaging starts at some future time t_F ($t_0 < t_F < T$) and the option pay-out function is dependent on the future realized variance $I(t_F, T)$.

By linearity, at valuation time t, $t_0 \leq t \leq T$, the realized variance can be represented as follows:

$$I(t_0, T) = I(t_0, t) + I(t, T), \tag{7}$$

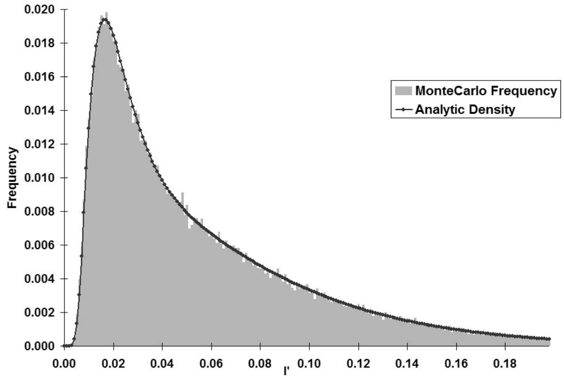
where $I(t_0, t)$ is the realized variance so far which is known at time t and $I(t, T)$ represents future realization. By pricing an existing volatility option, $I(t_0, t)$ is calculated using (4) based on known asset price fixings (in Section 7.1 we consider calculation of $I(t_0, t)$ in more details as it has important consequences for pricing and hedging). To suppress notations, we denote by $I(t)$ the state variable corresponding to the evolution of the realized variance $I(t, T)$ and by $I(0)$ the realized variance so far $I(t_0, t)$.

Using continuous-time quantities alleviates the analysis and allows us to obtain closed-form solutions for the pricing problem. We also see that if the asset price dynamics follow a jump-diffusion process with stochastic variance then the realized variance $I(t)$ is a stochastic process itself. In particular, it represents a process with a stochastic drift, no diffusion part, and a jump component.

Importantly, for practical purposes we need to investigate how well the continuous-time realized variance (5) approximates the actual realized variance computed by (4). We leave this analysis from our present discussion, and note that for daily fixings and longer maturities, the approximation turns out to be very accurate. As a single illustration, we apply Monte-Carlo simulation of the market model (1) using model parameters from the column titled "Implied" in Table (1) with the asset jump parameters $\nu = -0.1$ and $\delta = 0.1$. We simulate $100,000$ paths of the evolution of the asset price on daily intervals with the total number of days (within one path) equal to 252. Then for each path we calculate the discrete one-year realized variance using formula (4) with $N = AF = 252$. Finally, in Figure (1) we plot the histogram, scaled to sum up to one, of these realizations (denoted by "MonteCarlo Frequency"), which represents "empirical" probability density function (PDF), along with the Green function of theoretical PDF of $I(t)$ with $T = 1$

(denoted by "Analytic Density"), which is computed by inverting (23). From Figure (1) we see that our theoretical continuous-time process for the realized variance $I(t)$ adequately describes the evolution of the actual realized variance computed by (4).

Figure 1: Histogram of simulated discrete realized variance versus theoretical density of continuous-time realized variance



2.3 Options on the Realized Variance

To generalize, we denote the pay-off function of an option on the annualized realized variance or volatility by $u(\hat{I}, \hat{K}, o)$, where $o = 1$ ($o = 1/2$) stands for an option on the realized variance (volatility) and $\hat{K}^2$ ($\hat{K}^1$) is the delivery price measured in variance (volatility) points. The market convention is to quote variance swaps using strike prices and cap levels measured in volatility points, so that by pricing options on the realized variance we have to be express them in variance units.

We consider some of the most common derivative options on realized variance, including:

1) Swap on the realized variance (volatility) with the following pay-off function at time T:

$$u^{swp}(\hat{I}, \hat{K}, o) = \hat{I}^o - \hat{K}^{2o}. \tag{8}$$

2) Swap on the realized variance (volatility) with cap $\hat{C}^{2o}$:

$$u^{swpc}(\hat{I}, \hat{K}, o) = \min\{\hat{I}^o, \hat{C}^{2o}\} - \hat{K}^{2o}. \tag{9}$$

3) Call option on the realized variance (volatility) swap:

$$u^{swpo}(\hat{I}, \hat{K}, o) = \max\{\hat{I}^o - \hat{K}^{2o}, 0\}. \tag{10}$$

4) Call option on the realized variance (volatility) swap with cap $\hat{C}^{2o}$, $\hat{C}^{2o} > \hat{K}^{2o}$:

$$u^{swpoc}(\hat{I}, \hat{K}, o) = \max\{\min\{\hat{I}^o, \hat{C}^{2o}\} - \hat{K}^{2o}, 0\}. \tag{11}$$

These payoff functions are further multiplied by the notional amount of the contract measured in currency units per variance or volatility point respectively. For brevity, we take the notional amount to be one unit.

The strike of a variance swap $\hat{K}$ at the swap inception time is typically chosen in the way so that the initial value of the variance swap is zero. This strike is also termed as the fair variance and it is typically computed by the decomposition formula given in Demeterfi *et al* (1999) from available or interpolated prices of European call and put options. We, however, note that the fair strike computed this way might not be a good estimate for a model-free estimate of the expected realized variance due to liquidity (not all strikes are liquid enough especially those remote from the at-the-money strike), transaction costs (trading across many strikes is expensive), and discontinuous dynamics of the asset price (sudden price jumps undermine the effectiveness of the static replication).

The market convention is to cap the realized variance when the underlying is a stock to avoid possible huge pay-outs in case the stock's issuer would default (we note that the realized variance defined by (4) grows to infinity as $S(t)$ goes to zero). The cap level is typically chosen to be the fair variance multiplied by a factor of 2.5 or 3. Pricing volatility options under the default risk is studied in Sepp (2007). Here, we assume that the underlying asset is default-free. Alternatively, we can model the default event by assigning a large value to the mean of return jump ν in the dynamics (1).

3 Analytical Solution by Generalized Fourier Transform

In this section we derive semi-analytical formulas for pricing variance options under the dynamics (1). Without the loss of generality we assume that model parameters are time-homogeneous.

First, we introduce the logarithmic asset price $X(t) = \ln S(t)$, and augment the dynamics (1) with the dynamics for the realized variance $I(t)$, obtained from (5). As a result, we obtain the following joint dynamics of the state variables under the pricing measure $\mathbb{Q}$:

$$\begin{cases} dX(t) &= (r(t) - d(t) - \gamma m^j - \frac{1}{2}V(t))dt + \sqrt{V(t)}dW^s(t) + J^s dN(t),\ X(0) = \ln S, \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dW^v(t) + J^v dN(t),\ V(0) = V, \\ dI(t) &= V(t)dt + (J^s)^2 dN(t),\ I(0) = I. \end{cases} \tag{12}$$

The domain of definition of these variables is as follows: $X \in (-\infty, \infty)$, $V \in [0, \infty)$, $I \in [0, \infty)$. For brevity, whenever we write down a PIDE we assume that all boundaries are natural, which means that no explicit specification of boundary conditions is necessary, because for implementation of our analytical methods this analysis is complimentary. Relevant details on specifying boundary conditions by pricing volatility options can be found, for example, in Little-Pan (2001), Windcliff *et al* (2006), and Broadie-Jain (2008).

3.1 Pricing Equation

We introduce the time-to-maturity variable τ, $\tau = T-t$, and denote by $U(\tau, X, V, I, K, o)$ the forward value of the option on the de-annualized realized variance $I(t)$ with the pay-off function $u(I, K, o)$, where K is the de-annualized delivery price, $K = (T-t_0)^o \hat{K}^{2o}$, and (if applicable) with the de-annualized cap level $C = (T-t_0)^o \hat{C}^{2o}$.

Applying the risk-neutral pricing and Feynman-Kac formula for the market model (12), we obtain the following equation for the value function $U(\tau, X, V, I, K, o)$:

$$\begin{aligned}
& -U_\tau + (r(t) - d(t) - \gamma m^j - \frac{1}{2}V)U_X + \kappa(\theta - V)U_V + VU_I \\
& + \rho\varepsilon V U_{XV} + \frac{1}{2}VU_{XX} + \frac{1}{2}\varepsilon^2 V U_{VV} \\
& + \gamma \int_0^\infty \int_{-\infty}^\infty [U(X+J^s, V+J^v, I+(J^s)^2) - U]\varpi^v(J^v)\varpi^s(J^s)dJ^s dJ^v = 0, \\
& U(0, X, V, I, K, o) = u\left(I, K, o\right).
\end{aligned} \tag{13}$$

Since the equation (13) is linear in I, once $U(\tau, S, V, I, K, o)$ is found we retrieve the value of the option on the annualized realized variance $\hat{I}$ by normalizing and discounting:

$$\hat{U}(\tau, X, V, \hat{I}, \hat{K}, o) = \frac{D(\tau)}{(T-t_0)^o} U(\tau, X, V, I, K, o), \tag{14}$$

where $D(\tau)$ is appropriate discount factor.

To develop general approach for pricing options on the realized variance and volatility, we find the Green function (probability density function) corresponding to Eq.(13) and supplied with appropriate initial condition. Since the final condition for options on the realized variance is independent of the log-spot price X, we exclude this variable from the analysis.

As a result, the Green function of I, $G^I(\tau, V, I, I')$, corresponding to (13), solves the following backward Kolmogoroff equation:

$$\begin{aligned}
& -G^I_\tau + \kappa(\theta - V)G^I_V + VG^I_I + \frac{1}{2}\varepsilon^2 V G^I_{VV} \\
& + \gamma \int_0^\infty \int_{-\infty}^\infty [G^I(V+J^v, I+(J^s)^2) - G^I]\varpi^v(J^v)\varpi^s(J^s)dJ^s dJ^v = 0, \\
& G^I(0, V, I, I') = \delta(I - I'),
\end{aligned} \tag{15}$$

where $\delta(x)$ is the Dirac delta function.

Once the Green function is found we compute option value by calculating its expected payoff under the measure $\mathbb{Q}$:

$$U(\tau, X, V, I, K, o) = \int_0^\infty G^I(\tau, V, I, I') u\left(I', K, o\right) dI'. \tag{16}$$

First we solve for the Green function $G^I(\tau, V, I, I')$ and then we develop the pricing method that is more numerically efficient than computing convolution (16).

3.2 Solution to the Green Function

Solution for the Green function associated with equation (15) was obtained by Lipton (2001) through Fourier transform with respect to I'. Here we extend his result to the Heston model with price and volatility jumps and solve this equation by means of the generalized complex-valued Fourier transform. We note that there is a considerable advantage to solve PIDE (15) by means of this transform since it allows us to obtain analytical solutions for values of volatility options without the need to numerically evaluate the integral in Eq.(16).

We define the generalized forward, $\mathcal{F}_-$, and inverse Fourier transform, $\mathcal{F}_+$, with transform variable $\Psi = \Psi_R + i\Psi_I$, where $i = \sqrt{-1}$ and Ψ_R, $\Psi_I \in \mathbb{R}$, respectively as follows:

$$\begin{aligned} \widehat{U}(\Psi) &:= \mathcal{F}_-[U(x)](\Psi) = \int_{-\infty}^{\infty} e^{-\Psi x} U(x) dx, \\ \mathcal{F}_+[U(\Psi)](x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\Psi x} U(\Psi) d\Psi_I, \end{aligned} \tag{17}$$

In the above definition (17), $e^{-\Psi x}U(x)$ is assumed to be absolutely integrable as a function of Ψ_R. The range of possible values for Ψ_R is determined by the problem at hand: for the transformed initial condition of the Green's function Ψ_R is unrestricted, while for transforms of the payoff functions considered here we need to take $\Psi_R < 0$ to make the pricing problem well-defined. Representation (17) is a slightly modified version of that introduced by Carr-Madan (1999) and Lewis (2001) and it is written in terms of transform variable $\Psi \in \mathbb{C}$, which simplifies notations and formulas.

We informally extend the domain of definition of I' to be $(-\infty, \infty)$ and introduce the Fourier-transformed Green function $\widehat{G}^I(\tau, V, I, \Psi)$ by applying transform (17) with respect to the variable I' to the Green function, $G^I(\tau, V, I, I')$:

$$\widehat{G}^I(\tau, V, I, \Psi) = \mathcal{F}_-[G^I(I')](\Psi). \tag{18}$$

We then obtain the following problem for $\widehat{G}^I(\tau, V, I, \Psi)$:

$$\begin{aligned} &- \widehat{G}^I_\tau + \kappa(\theta - V)\widehat{G}^I_V + \frac{1}{2}\varepsilon^2 V \widehat{G}^I_{VV} + V\widehat{G}^I_I \\ &+ \gamma \int_0^\infty \int_{-\infty}^\infty [\widehat{G}^I(V + J^v, I + (J^s)^2) - \widehat{G}^I] \varpi^v(J^v)\varpi^s(J^s) dJ^s dJ^v = 0, \\ &\widehat{G}^I(0, V, I, \Psi) = e^{-\Psi I}. \end{aligned} \tag{19}$$

In Appendix A1, we show that the solution to this problem is given by:

$$\widehat{G}^I(\tau, V, I, \Psi) = e^{-\Psi I + A^I(\tau,\Psi) + B^I(\tau,\Psi)V + \Gamma^I(\tau,\Psi)}, \tag{20}$$

where

$$
\begin{aligned}
A^I(\tau,\Psi) &= -\frac{\kappa\theta}{\varepsilon^2}\left[\psi_+\tau + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta\tau}}{2\zeta}\right)\right],\\
B^I(\tau,\Psi) &= -2\Psi\frac{1-e^{-\zeta\tau}}{\psi_- + \psi_+ e^{-\zeta\tau}},\\
\Gamma^I(\tau,\Psi) &= \gamma\left(\frac{\exp\left\{\frac{-\nu^2\Psi}{2\Psi\delta^2+1}\right\}}{\sqrt{2\Psi\delta^2+1}}\right)\frac{-2\eta\ln\left[\frac{\gamma_- + \gamma_+ e^{-\zeta\tau}}{2\zeta}\right] + (\varepsilon^2 - 2\eta\kappa - \eta\psi_+)\tau}{\varepsilon^2 - 2\eta\kappa - 2\eta^2\Psi} - \gamma\tau,\\
\psi_\pm &= \mp\kappa + \zeta,\ \zeta = \sqrt{\kappa^2 + 2\varepsilon^2\Psi},\ \gamma_\pm = \psi_\pm \mp 2\eta\Psi,\ \Psi_R > -1/(2\delta^2).
\end{aligned}
\tag{21}
$$

Detailed analysis of two complex-valued logarithms in (21) by Sepp (2007) shows that they are continuous as functions of Ψ provided:

$$
\Psi_R > \max\left\{-\frac{\kappa}{2\eta}, -\frac{\kappa}{2\varepsilon^2}\right\}. \tag{22}
$$

As a result, we evaluate $G(\tau,V,I,I')$ by computing the inverse transform:

$$
\begin{aligned}
G^I(\tau,V,I,I') = \mathcal{F}_+[\widehat{G}^I(\Psi)](I') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\Psi I'}\widehat{G}^I(\tau,V,I,\Psi)d\Psi_I\\
&= \frac{1}{\pi}\int_0^{\infty}\Re\left[e^{\Psi I'}\widehat{G}^I(\tau,V,I,\Psi)\right]d\Psi_I.
\end{aligned}
\tag{23}
$$

This inversion can be done by using numerical integration routines or by means of Fast Fourier Transform (FFT) method. Once the values of $G^I(\tau,V,I,I')$ are computed for the entire state space of I', we can calculate the value of the claim by evaluating integral (16) numerically.

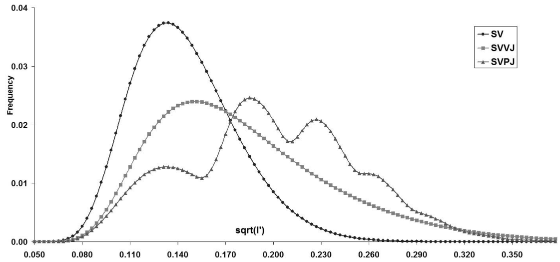
To make sure that formula (23) makes sense we study the limiting behavior of the kernel (20). To be concrete, we take $\Psi = ih$ with $h\in\mathbb{R}$, and compute the limit:

$$
\lim_{h\to\pm\infty}\Re\left[G^I(\tau,V,I,ih)\right] \sim e^{-\frac{V+\kappa\theta\tau}{\varepsilon}\sqrt{|h|}}. \tag{24}
$$

Accordingly, the integrand in formula (23) is a decaying function of Ψ and the integral is finite. Limit (24) can serve as a mean to obtain a fixed integration bound corresponding to a desired level of accuracy. We point out that the convergence might be slow for small values of τ and initial variance V.

In Figure (2), we plot the density function of the square root of the realized variance, $\sqrt{I(t)}$, with $\tau = 1$ using model parameters from the column titled "Implied" in Table (1) that corresponds to three models: 1) SV model with $\gamma = 0$, 2) SVVJ model with variance jumps and no price jumps, 3) SVPJ model with no variance jumps and with price jump parameters $\nu = -0.1378$ and $\delta = 0.0$ (for this choice of ν, the expected realized variance in the SVVJ model is approximately equal to that in the SVPJ model). We see that the density of the realized variance in the SVVJ model becomes multi-modal as the reflection of the discontinuous dynamics of $I(t)$. We also see the Heston model with either variance or price jumps implies heavy right tails of the realized variance compared to the Heston model with no jumps. Finally, we see that the density function of realized variance in the Heston model with no jumps resembles the density of a log-normal random variable (we employ this fact later to obtain approximate solutions for volatility options).

Figure 2: Density of the square root of the realized variance



3.3 Applications

To reduce the computation burden of using pricing formula (16) directly, we now consider the pay-off functions of volatility options reported in Section 2.3 and find their respective Fourier-transformed functions so that option values can be computed by doing a single numerical inversion.

First we start with value function of the realized variance swap for which the analytical solution is given by:

$$\begin{aligned} U^{swp}(\tau, X, V, I, K, 1) = I - K + \frac{\theta}{\kappa}\left(\tau\kappa + e^{-\kappa\tau} - 1\right) + \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right) V \\ + \frac{\gamma\eta}{\kappa^2}\left(\tau\kappa + e^{-\kappa\tau} - 1\right) + \gamma\tau(\nu^2 + \delta^2). \end{aligned} \quad (25)$$

Expanding $U^{swp}(\tau, X, V, I, K, 1)$ in Taylor series around $\tau = 0$, we obtain:

$$U^{swp}(\tau, X, V, I, K, 1) = I - K + (V + \gamma(\nu^2 + \delta^2))\tau + \frac{1}{2}\kappa(\theta - V)\tau^2 + \frac{1}{2}\eta\gamma\tau^2 + O(\tau^3). \quad (26)$$

Expansion (26) allows to analyze the behavior of the expected realized variance in the short term. We see that the contribution of order τ to the realized variance I is produced by the instantaneous variance V and the expected realization of the price jumps while the contribution of all remaining parameters is of order τ^2 or less. We also see that the first term of order τ^2 represents the effect of mean-reversion: it is positive if $V < \theta$ and negative otherwise, while the second term reflects the impact of the variance jumps, which is always positive. We also see that all being the same, Heston model with jumps implies higher prices for swaps on the realized variance.

For the volatility option with the value function $U(\tau, X, V, I, K, o)$ and the non-linear payoff function $u\left(I', K, o\right)$, we use pricing formula (16) along with formula for the Green function $G^I(\tau, V, I, I')$ in (23) to obtain:

$$
\begin{aligned}
&U(\tau, X, V, I, K, o) = \int_0^\infty G^I(\tau, V, I, I') u\left(I', K, o\right) dI' \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{\Psi I'} \widehat{G}^I(\tau, V, I, \Psi) u\left(I', K, o\right) d\Psi_I dI' \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_0^\infty e^{\Psi I'} u\left(I', K, o\right) dI' \right] \widehat{G}^I(\tau, V, I, \Psi) d\Psi_I \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \widehat{G}^I(\tau, V, I, \Psi) \check{u}(\Psi, K, o) d\Psi_I = \frac{1}{\pi} \int_0^\infty \Re \left[\widehat{G}^I(\tau, V, I, \Psi) \check{u}(\Psi, K, o) \right] d\Psi_I,
\end{aligned}
\tag{27}
$$

where $\check{u}(\Psi, K, o)$ is the inverse transform of the option pay-out function:

$$
\check{u}(\Psi, K, o) = \int_0^\infty e^{\Psi I'} u\left(I', K, o\right) dI', \tag{28}
$$

and to make valid the interchange of the integration order we require $\check{u}(\Psi, K, o)$ to be bounded as function of Ψ_R.

Calculating transform (28) for option payoff functions on the realized variance with $o = 1$, yields

$$
\begin{aligned}
\check{u}^{swp}(\Psi, K, 1) &= \frac{1}{\Psi^2} + \frac{K}{\Psi}, \\
\check{u}^{swpc}(\Psi, K, 1) &= \frac{1}{\Psi^2}\left(1 - e^{C\Psi}\right) + \frac{K}{\Psi}, \\
\check{u}^{swpo}(\Psi, K, 1) &= \frac{1}{\Psi^2} e^{K\Psi}, \\
\check{u}^{swpoc}(\Psi, K, 1) &= \frac{1}{\Psi^2}\left(e^{K\Psi} - e^{C\Psi}\right) - \frac{e^{C\Psi}(C - K)}{\Psi},
\end{aligned}
\tag{29}
$$

provided $\Psi_R < 0$. Now we are in position to price these options using formula (27).

We note that, although to value the variance swap no transform technique is necessary as the analytical solution is given by (25), inversion of the transformed pay-off function $\check{u}^{swp}(\Psi, K, 1)$ can help us to tune up our numerical inversion routines for pricing formula (27).

By analogy, for options on the realized volatility with $o = \frac{1}{2}$, we calculate the transformed payoffs (28) as follows. For the swap on the realized volatility, we obtain:

$$
\check{u}^{swp}(\Psi, K, 1/2) = \int_0^\infty e^{\Psi I'} [\sqrt{I'} - K] dI' = \frac{\sqrt{\pi}}{2\left(-\Psi\right)^{3/2}} + \frac{K}{\Psi}, \tag{30}
$$

provided $\Psi_R < 0$.

The transform function of the pay-off function of the call option on the volatility swap is given by:

$$\breve{u}^{swpo}(\Psi, K, 1/2) = \int_0^\infty e^{\Psi I'}[\max\{\sqrt{I'} - K, 0\}]dI' = \frac{\sqrt{\pi}\left(1 - \mathrm{erf}(K\sqrt{-\Psi})\right)}{2\left(-\Psi\right)^{3/2}}, \quad (31)$$

where $\Psi_R < 0$ and $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$ is the error function of a complex-valued argument (to evaluate this function we employ the algorithm (7.1.29) from Abramowitz-Stegun (1972)).

For the pay-off functions of volatility swap with cap and capped call option, we obtain the following formulas:

$$\begin{aligned} \breve{u}^{swpc}(\Psi, K, 1/2) &= \frac{\sqrt{\pi}\mathrm{erf}(C\sqrt{-\Psi})}{2\left(-\Psi\right)^{3/2}} + \frac{K}{\Psi}, \\ \breve{u}^{swpoc}(\Psi, K, 1/2) &= \frac{\sqrt{\pi}\left(\mathrm{erf}(C\sqrt{-\Psi}) - \mathrm{erf}(K\sqrt{-\Psi})\right)}{2\left(-\Psi\right)^{3/2}}. \end{aligned} \quad (32)$$

Now we are able to compute the prices of volatility options by inverting integral in (27). A robust choice of Ψ_R for numerical evaluation of formula (27) is $\Psi_R = -1$.

4 Pricing Forward-Start Volatility Options

In this section we develop a general approach for pricing forward-start options on the realized variance and apply it for pricing forward-start options under the dynamics (1).

4.1 General Valuation Approach

In general, a forward start option derives its value from two future observations of the underlying variable at times t_F and T, $t < t_F < T$. At time $t = t_F$, a fixing is made and the forward-start option becomes a standard option with only one underlying variable. Accordingly, our main issue is to compute values of these contracts at time t, $t < t_F$. We introduce annualized time-to-fixing τ_F, $\tau_F = t_F - t$, $\tau_F > 0$, and annualized option tenor time τ_T, $\tau_T = T - t_F$. We note that in the present context:

$$\hat{I}(t_F, T) = \frac{1}{T - t_F} I(t_F, T). \quad (33)$$

We denote by $\hat{U}(\tau_F, \tau_T, V, \hat{I}, \hat{K}, o)$ the value function of the forward start option on $\hat{I}(t_F, T)$ and by $u(\hat{I}, \hat{K}, o)$ its payoff function (it can be any of the pay-off functions considered in Section 2.3). We are interested in calculating the fair value of this forward start option at time t_0, $t_0 < t_F < T$.

Employing the risk-neutral pricing under the measure $\mathbb{Q}$, we can present the value of the forward-start option at time $t = t_0$ as follows:

$$\hat{U}(\tau_F, \tau_T, V, \hat{I}, \hat{K}, o) = \frac{D(\tau_F)}{(\tau_T)^o} \int_0^\infty \int_0^\infty Z(\tau_T, V', I', K, o) G^{vI}(t_0, I, V; \tau_F, I', V') dI' dV', \tag{34}$$

where Z is the option value on the de-annualized realized variance:

$$Z(\tau_T, V', I', K, o) = D(\tau_T) \int_0^\infty \int_0^\infty u(I'', K, o) G^{vI}(\tau_F, V', I'; \tau_T, V'', I'') dI'' dV'', \tag{35}$$

and $G^{vI}(t_0, V, I; \tau_F, V', I')$ is the transition density function of the joint evolution of $(V(t), I(t))$, $t > t_0$, solving backward PIDE (15) subject to initial condition:

$$G^{vI}(t_0, V, I; t_0, V', I') = \delta(I - I')\delta(V - V').$$

Using the fact that the payoff function does not depend on $V(\tau_T)$ and that fixing starts only at time t_F, that is $I(t) = 0$ at $t = t_F$, and finally taking into account that

$$\int_0^\infty G^{vI}(t_0, V, I; \tau_F, V', I') dV' = G^I(\tau_F, V, I, I'),$$

we simplify Eq(34) to obtain:

$$\hat{U}(\tau_F, \tau_T, V, \hat{I}, \hat{K}, o) = \frac{D(\tau_F)}{(\tau_T)^o} \int_0^\infty Z(\tau_T, V', 0, K, o) G^v(\tau_F, V, V') dV', \tag{36}$$

where $Z(\tau_T, V, I, K, o)$ is the value function of the volatility option with maturity time τ_T:

$$Z(\tau_T, V, I, K, o) = D(\tau_T) \int_0^\infty u(I', K, o) G^I(\tau_T, V, I, I') dI', \tag{37}$$

and $G^v(\tau_F, V, V')$ is the transition density of the variance $V(t)$.

Accordingly, formula (36) implies that the value of the forward-start option can be represented as the expected value of the underlying option on the realized variance Z with inception at time τ_F and maturity time τ_T, where expectation is taken at valuation time t over all possible states of the variance $V(\tau_F)$ at fixing time τ_F.

Our task is simplified since the closed-form solution for the option value $Z(\tau_T, V', I', K, o)$ and the Green function of the variance $G^v(\tau_F, V, V')$ are available in closed-form under the dynamics (1). Here, we also assume that the Fourier transform of $G^v(\tau_F, V, V')$ denoted by $\widehat{G}^v(\tau_F, V, \Theta)$ is known and given by (41), which we derive in the next section.

Furthermore, by applying our integral pricing formula formula (27), we simplify our general valuation formula (36) as follows:

$$
\begin{aligned}
\hat{U}(\tau_F,\tau_T,V,\hat{I},\hat{K},o) &= \frac{D(\tau)}{2\pi\tau_T^o}\int_0^\infty\int_{-\infty}^\infty G^I(\tau_T,V',I,\Psi)\check{u}(\Psi,K,o)G^v(\tau_F,V,V')d\Psi_I dV' \\
&= \frac{D(\tau)}{2\pi\tau_T^o}\int_{-\infty}^\infty e^{-\Psi I+A^I(\tau_T,\Psi)+\Gamma^I(\tau_T,\Psi)}\check{u}(\Psi,K,o)\left[\int_0^\infty e^{B^I(\tau_T,\Psi)V'}G^v(\tau_F,V,V')dV'\right]d\Psi_I \\
&= \frac{D(\tau)}{2\pi\tau_T^o}\int_{-\infty}^\infty e^{-\Psi I+A^I(\tau_T,\Psi)+\Gamma^I(\tau_T,\Psi)}\widehat{G}^v(\tau_F,V,-B^I(\tau_T,\Psi))\check{u}(\Psi,K,o)d\Psi_I,
\end{aligned}
\tag{38}
$$

with $D(\tau)=D(\tau_F)D(\tau_T)$ and where in the first line we use pricing formula (27) for the option value function Z, in the second line we employ formula (20) for the Green function G^I and exchange the order of integration (which is valid since the integrands are assumed to be bounded functions). In the third line, we first note that the expression in square brackets in the second line is the moment generating function of the variance given by formula (41) and then we use the latter formula to obtain the final result.

Formula (38) is one of our key results which allows us to price forward-start options by inverting a single one-dimensional integral. As a result, for pricing forward-start options we use formula (38), where $\check{u}(\Psi,K,o)$ is the transformed payoff of the underlying option defined by (28). The integrand in the last line in (28) can be considered as the kernel of Green's function of the realized future variance realized over period $[\tau_F,T]$.

In Section 5, we consider another important example of forward-start options on the square root of future realized variance over $[\tau_F,T]$ as seen at time t, which are commonly called options on the VIX.

As we see, to solve the problem of pricing forward-start options efficiently, we need to find a closed-form solution for the density and moment generating function of the variance $V(t)$, which we do next.

4.2 Transition Density of the Variance

To implement pricing formula (36) and (38), we need to find an explicit expression for the Green function of the variance, $G^v(\tau,V,V')$, and its Fourier transform. $G^v(\tau,V,V')$ as function of V solves the backward Kolmogoroff equation:

$$
\begin{aligned}
&-G^v_\tau+\kappa(\theta-V)G^v_V+\frac{1}{2}\varepsilon^2 V G^v_{VV}+\gamma\int_0^\infty[G^v(V+J^v)-G^v(V)]\varpi^v(J^v)dJ^v=0,\\
&G^v(0,V,V')=\delta(V-V').
\end{aligned}
\tag{39}
$$

In Appendix A2 we show how to obtain the solution to this PIDE by applying generalized Fourier transform in V' with transform variable $\Theta=\Theta_R+i\Theta_I$:

$$
G^v(\tau,V,V')=\frac{1}{2\pi}\int_{-\infty}^\infty e^{\Theta V'}\widehat{G}^v(\tau,V,\Theta)d\Theta_I=\frac{1}{\pi}\int_0^\infty\Re\left[e^{\Theta V'}\widehat{G}^v(\tau,V,\Theta)\right]d\Theta_I, \tag{40}
$$

where

$$\widehat{G}^v(\tau, V, \Theta) = e^{A^v(\tau,\Theta)+B^v(\tau,\Theta)V+\Gamma^v(\tau,\Theta)}, \tag{41}$$

and

$$\begin{aligned}
A^v(\tau, \Theta) &= -\frac{2\kappa\theta}{\varepsilon^2}\ln\left(\frac{\varepsilon^2}{2\kappa}\Theta\left(1-e^{-\kappa\tau}\right)+1\right), \\
B^v(\tau, \Theta) &= -\frac{\Theta e^{-\kappa\tau}}{\frac{\varepsilon^2}{2\kappa}\Theta\left(1-e^{-\kappa\tau}\right)+1}, \\
\Gamma^v(\tau, \Theta) &= \frac{2\eta\gamma}{\varepsilon^2-2\eta\kappa}\ln\left(\frac{2\kappa(1+\eta\Theta)}{\varepsilon^2\Theta+2\kappa-\Theta e^{-\kappa\tau}(\varepsilon^2-2\eta\kappa)}\right).
\end{aligned} \tag{42}$$

Sepp (2007) shows that two complex-valued logarithms in (42) are continuous as functions of Θ provided:

$$\Theta_R > \max\left\{-\frac{1}{\eta}, -\frac{2\kappa}{\varepsilon^2}\right\}. \tag{43}$$

Next we check that formula (40) for $G^v(\tau, V, V')$ is well-defined by assuming that $\Theta = ip$ and taking the limits in (41) to obtain:

$$\lim_{p\to\pm\infty}\Re\left[\widehat{G}^v(\tau, V, ip)\right] \sim e^{-\frac{2\kappa\theta}{\varepsilon^2}\ln\left(\frac{\varepsilon^2}{2\kappa}(1-e^{-\kappa\tau})|p|\right)}. \tag{44}$$

Accordingly, the integral in formula (40) makes sense and we can use limit (44) to establish fixed integration bounds. A fast way to compute $G^v(\tau, V, V')$ for the entire state space of V' is to apply FFT.

For our subsequent development, we also find the steady-state density of the variance $G^{v\infty}(V')$ and, in particular, its Fourier transform $\widehat{G}^{v\infty}(V')$. We obtain solution for $\widehat{G}^{v\infty}(V')$ from $\widehat{G}^v(\tau, V, V')$ by letting $\tau \to \infty$:

$$\widehat{G}^{v\infty}(\Theta) = e^{A^{v\infty}(\Theta)+\Gamma^{v\infty}(\Theta)}, \tag{45}$$

and

$$\begin{aligned}
A^{v\infty}(\Theta) &= -\frac{2\kappa\theta}{\varepsilon^2}\ln\left(\frac{\varepsilon^2}{2\kappa}\Theta+1\right), \\
\Gamma^{v\infty}(\Theta) &= \frac{2\eta\gamma}{\varepsilon^2-2\eta\kappa}\ln\left(\frac{2\kappa(1+\eta\Theta)}{\varepsilon^2\Theta+2\kappa}\right).
\end{aligned} \tag{46}$$

Finally, we compute the expected value of the variance, $\overline{V}(\tau)$, and its variance, $\mathbb{V}[V(\tau)]$. They can be computed by means of row moments which are obtained using the moment generating function of V given by formula (41):

$$\mathbb{M}_k[V(\tau)] = (-1)^k\frac{\partial^k}{\partial\Theta_R^k}\widehat{G}^v(\tau, V, \Theta_R)\mid_{\Theta_R=0,\Theta_I=0}. \tag{47}$$

Performing calculations for the Heston model with jumps in the variance, we obtain the following formulas for $\bar{V}$ and $\mathbb{V}[V(\tau)]$:

$$\begin{aligned}\overline{V}(\tau) &= e^{-\kappa\tau}V + \left(1 - e^{-\kappa\tau}\right)\theta + \left(1 - e^{-\kappa\tau}\right)\frac{\lambda\eta}{\kappa},\\ \mathbb{V}[V(\tau)] &= \frac{\varepsilon^2}{\kappa}\left(1 - e^{-\kappa\tau}\right)\left(e^{-\kappa\tau}V + \frac{1}{2}(1 - e^{-\kappa\tau})\theta\right) + \\ &+ \frac{\lambda\eta}{2\kappa^2}\left(1 - e^{-\kappa\tau}\right)\left(2\eta\kappa + \varepsilon^2 + (2\eta\kappa - \varepsilon^2)e^{-\kappa\tau}\right).\end{aligned} \tag{48}$$

From (48), we see that the long-term mean of the implied variance is $\theta + \frac{\lambda\eta}{\kappa}$ and the long-term variance of the implied variance is $\frac{\varepsilon^2\theta}{2\kappa} + \frac{\eta\lambda}{2\kappa^2}\left(2\eta\kappa + \varepsilon^2\right)$.

4.3 Convexity Adjustment Formula

An approximate pricing formula for forward-start volatility options can be obtained by expanding the value function of the underlying claim in formula (36) in Taylor series (we assume that the value function $Z(\tau_T, V, I, K, o)$ is sufficiently smooth as function of V) around the expected value of the variance $V(\tau_F)$ denoted by $\overline{V}$ (a similar idea is used by Lipton (2001) by pricing forward-start equity options):

$$\begin{aligned}\hat{U}(\tau_F, \tau_T, V, \hat{I}, \hat{K}, o) &= \frac{D(\tau)}{(\tau_T)^o}\int_0^\infty G^v(\tau_F, V, V')\times \\ &\left(Z(\tau_T, V, I, K, o)|_{V=\overline{V}} + \frac{\partial}{\partial V}Z(\tau_T, V, I, K, o)|_{V=\overline{V}}(V' - \overline{V})\right. \\ &\left. + \frac{1}{2}\frac{\partial^2}{\partial V^2}Z(\tau_T, V, I, K, o)|_{V=\overline{V}}(V' - \overline{V})^2 + ...\right) dV'.\end{aligned} \tag{49}$$

Computing expectations, we obtain:

$$\hat{U}(\tau_F, \tau_T, V, \hat{I}, \hat{K}, o) \approx \frac{D(\tau)}{(\tau_T)^o}\left(Z(\tau_T, \overline{V}(\tau_F), I, K, o) + \frac{1}{2}\mathbb{V}[V(\tau_F)]\frac{\partial^2}{\partial V^2}Z(\tau_T, \overline{V}(\tau_F), I, K, o)\right). \tag{50}$$

The advantage of using the convexity adjustment formula is that the second order partial derivative with respect to V in formula (50) can be computed by inverting the corresponding derivative of the option value function (23). This approximation is rough when the volatility of the variance is high, however it allows us to quantify the impact of the volatility of the variance on values of forward-start volatility options.

5 VIX Futures and Options

In this section we apply our methodology for pricing VIX futures and option contracts. Some additional details on modeling the VIX under the dynamics (1) can be found in Sepp (2008). For first time, we propose a new method to estimate parameters of the model (1) from time series of the VIX.

5.1 The VIX

The VIX stands for Chicago Board Options Exchange (CBOE) Volatility Index which measures the implied volatility of S&P500 stock index options with maturity 30 days. In broad sense, the VIX represents the market's expectation of the annualized volatility of the S&P500 index over the next 30 day period. By now, investors can trade with exchange-listed VIX futures contracts, which began trading in 2004, and VIX option contracts, which began trading in February 2006.

The VIX is computed by CBOE every minute using liquid market quotes of short-term call and put options on the S&P500 index. More details on the VIX calculation can be found in the VIX white paper (CBOE 2006). Here, we model the VIX by assuming that the variance of returns on the S&P500 index is driven by dynamics (1) and utilizing our developed methodology. We note that is the case of VIX futures contracts we deal with continuous time variable so that there is no discretization error (as a matter of fact, we take $AF = 365$ and $N = 30$). According to the market convention, the VIX and the VIX futures are scaled by the factor of 100, which we omit for brevity.

The spot value of the VIX at time t, denoted by $F(t)$, measures the square root of the expected annualized realized variance for options with tenor $\tau_T = \frac{30}{365}$:

$$F(t) = \sqrt{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tau_T}\int_t^{t+\tau_T} I(t')dt' \mid \mathcal{F}(t)\right]}, \tag{51}$$

where filtration $\mathcal{F}(t)$ contains all information available at time t.

As a result, the futures value of the VIX, $F(t,T)$, $T > t$, is the expectation of the VIX at time T:

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tau_T}\int_T^{T+\tau_T} I(t')dt' \mid \mathcal{F}(T)\right]} \mid \mathcal{F}(t)\right]. \tag{52}$$

Our task consists in pricing options on the VIX and to solve this problem effectively we need to find the transition density function of $F(t)$. To accomplish this purpose, we use formula (25) for the expected future realized variance over the period τ_T under model (1) to obtain:

$$F(t) = \sqrt{U^{swp}(\tau_T, X, V, 0, 0, 1)} = \sqrt{A^F + B^F V(t)}, \tag{53}$$

where

$$\begin{aligned} A^F &= \frac{1}{\tau_T}\left(\frac{\theta}{\kappa}\left(\tau_T\kappa + e^{-\kappa\tau_T} - 1\right) + \frac{\gamma\eta}{\kappa^2}\left(\tau_T\kappa + e^{-\kappa\tau_T} - 1\right)\right) + \gamma(\nu^2 + \delta^2), \\ B^F &= \frac{1}{\tau_T}\frac{1}{\kappa}\left(1 - e^{-\kappa\tau_T}\right). \end{aligned} \tag{54}$$

From (53) we see that a derivative security on the VIX is essentially a bet on the future variance. As a result, our task is simplified since we know the transition density of the variance.

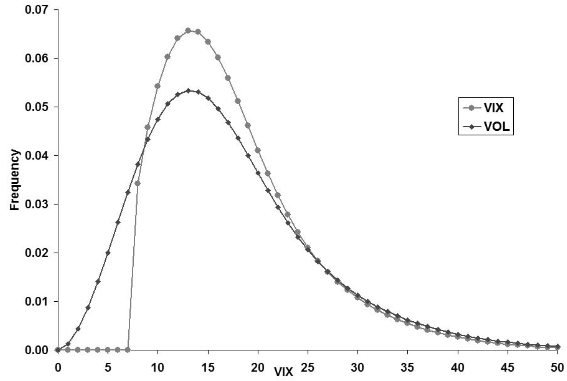
Since the spot VIX is a function of $V(t)$, we can also find the transition density function of $F(t)$, denoted by $G^F(\tau, V, F')$, using the transition density function of the implied variance, $G^v(\tau, V, V')$, as follows:

$$G^F(\tau, V, F') = \frac{2F'}{B^F} G^v\left(\tau, V, \frac{(F')^2 - A^F}{B^F}\right) \mathbf{1}_{\{(F')^2 \geq A^F\}}, \tag{55}$$

and employ it in our valuations.

In Figure (3), we show the probability density function of the volatility, $\sqrt{V(t)}$, and the VIX spot value, $F(t)$, using model parameters from column titled "Implied" in Table (1). We see that in some extent $F(t)$ corresponds to the shifted processes of the volatility. Noticeably, $F(t)$ has a limited down-side potential while the likelihood of up-side values is equivalent to the original processes. Accordingly, a short position in the VIX futures contracts is associated with rather limited gains and virtually unlimited losses.

Figure 3: The density function of the volatility, $\sqrt{V(\tau)}$, (Vol) and the spot VIX, $F(\tau)$, (VIX) with $\tau = 0.5$ (the x-axes is scaled by the factor of 100)



5.2 Pricing options on the VIX

Let $U(\tau, V, K)$ denote the value of the option on the VIX spot value $F(\tau)$ at time τ with payoff function $u(F, K)$ and strike price K measured per volatility point. We use our general pricing formula (36) for forward-start options along with formula (53) to get:

$$U(\tau, V, K) = D(\tau) \int_0^\infty G^v(\tau, V, V') \left[u(\sqrt{A^F + B^F V'}, K)\right] dV'. \tag{56}$$

We note that no discounting is applied for time $(\tau, \tau + \tau_T)$ since the payoff occurs at time τ, and no discounting is used for pricing VIX futures contracts.

Formula (56) was independently obtained by Zhang-Zhu (2006) and Zhu-Zhang (2007). Here, to avoid the need for numerical integration in formula (56), we develop an alternative valuation method based on the generalized Fourier transform. We note that, since the VIX squared $F^2(t)$ is a linear function of the variance $V(t)$, we can use the Fourier transform of the variance, $\widehat{G}^v(\tau, t_F, V, \Theta)$, given by (41) to obtain the Fourier transform of the Green function of $F^2(t)$, denoted by $\widehat{G}^{F^2}(\tau, V, \Theta)$, in the following way:

$$\widehat{G}^{F^2}(\tau, V, \Theta) = e^{-A^F\Theta}\widehat{G}^v(\tau, V, B^F\Theta). \tag{57}$$

As a result, we can modify our general pricing formula (27) along with the Fourier transform (57) to get an alternative pricing formula for options on the VIX as follows:

$$\begin{aligned} U(\tau, V, K) =& \frac{D(\tau)}{2\pi}\int_{-\infty}^{\infty} \widehat{G}^{F^2}(\tau, V, \Theta)\check{u}(\Theta, K)d\Theta_I \\ =& \frac{D(\tau)}{\pi}\int_{0}^{\infty} \Re\left[\widehat{G}^{F^2}(\tau, V, \Theta)\check{u}(\Theta, K)d\Theta_I\right], \end{aligned} \tag{58}$$

where $\check{u}(\Theta, K)$ is the inverse transform of the pay-out function expressed as function of $F^2(t)$:

$$\check{u}(\Theta, K) = \int_0^\infty e^{\Theta(F^2)'}u(\sqrt{(F^2)'}, K)d(F^2)'. \tag{59}$$

In Section 3.3 we have derived explicit transforms of payoffs, $\check{u}(\Theta, K, 1/2)$, for volatility swap given by (30) (a swap with zero strike corresponds to the VIX futures contract), for volatility call option given by (31) (which now corresponds to the call option on the VIX). In the present setting, we use pricing formula (58) along with the respective transform to value derivatives on the VIX.

5.3 Arbitrage bounds

Finally, we discuss a not-so-obvious distinction between the value of the forward-start volatility swap, the square root of the value of the forward-start variance swap, and the value of the VIX futures contract. For brevity, we assume that the strike price is zero and no discounting factor is applied. We assume that the time-to-fixing is τ_F and the maturity time is T, $T > \tau_F$.

In case of the forward-start volatility swap, we deal with expectation of the form:

$$\hat{U}^{FSVOL} = \mathbb{E}^{V(\tau_F), I(\tau_F, T)}\left[\sqrt{\frac{1}{T-\tau_F}I(\tau_F, T)}\right],$$

where $\mathbb{E}^{V(\tau_F), I(\tau_F, T)}$ denotes the time-t expectation with respect to future values of $V(\tau_F)$ and $I(\tau_F, T)$.

For the square root of the value of the forward-start variance swap we have:

$$\sqrt{\hat{U}^{FSVAR}} = \sqrt{\mathbb{E}^{V(\tau_F), I(\tau_F, T)}\left[\frac{1}{T-\tau_F}I(\tau_F, T)\right]}.$$

Finally, in case of the VIX futures contract we deal with expectation of the following form:

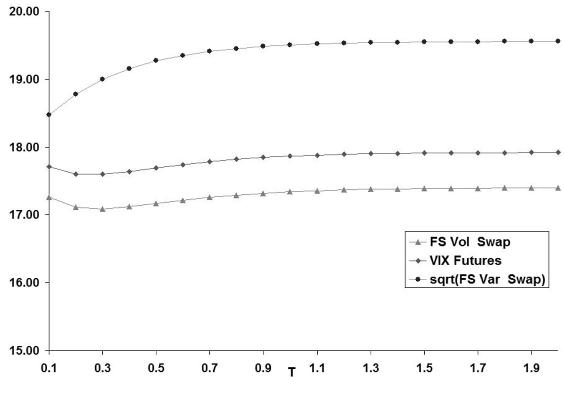
$$U^{VIX} = \mathbb{E}^{V(\tau_F)}\left[\sqrt{\mathbb{E}^{I(\tau_F,T)}\left[\frac{1}{T-\tau_F}I(\tau_F,T)\right]}\right].$$

Since the square root is a concave function, by Jensen inequality we have:

$$\hat{U}^{FSVOL} \leq U^{VIX} \leq \sqrt{\hat{U}^{FSVAR}}. \tag{60}$$

Accordingly, the value of the VIX futures contract is more expensive than that of the forward-start volatility swap, while the square root of the forward-start variance swap is more expensive than the VIX futures contract. In Figure (4), we show the term structure of these contracts' values as functions of T with $\tau_F = T - 1/12$ calculated by using model parameters from column titled "Implied" in Table (1). In general, the observed differences are caused by the convexity effect and they are roughly proportional to the square of volatility of volatility parameter. In trivial case of the constant variance, inequality (60) becomes equality.

Figure 4: The term structure of the forward-start volatility swap (FS Vol Swap), the VIX futures contract (VIX Futures), and the square root of the forward-start variance swap (sqrt(FS Var Swap)) as functions of T with $\tau_F = T - 1/12$



5.4 Model Estimation using VIX data

Now we apply our developed analytic tools to estimate parameters of the model (1) from both historical and market data of VIX options. We note that the VIX data provides a clear indication of the implied variance jump premium as the VIX dynamics are associated with the dynamics of the S&P500 index volatility. However, the VIX might not be a good indicator of the price jump premium related to in the S&P500. To make our estimation robust, we assume only variance jumps in the proposed market dynamics (1).

5.4.1 Historical Estimation

The VIX dynamics provides a valuable information about the implied evolution of the volatility dynamics of the S&P500 index. By using the VIX ime series data, Dotsis *et al* (2007) performed estimation of the Heston stochastic variance model with different jump assumptions by applying the maximum likelihood (ML) method. Although powerful, the ML method might be poor in fitting the tails of the empirical distribution, which is very important to appropriately estimate variance jump parameters. We develop an alternative method which can directly be applied to fitting the distribution tails and easily be visualized. We only assume that the time series is stationary and ergodic thus the time distribution of the sample is equivalent (in probability) to the spacial distribution of this sample.

By Eq. (53) we obtain that the VIX squared, $F^2(t)$, is the linear function of the variance:

$$F^2(t) = A^F + B^F V(t). \tag{61}$$

Now we assume that the variance $V(t)$ has its steady-state distribution and we compute the Fourier transform of the steady-state density function of F^2, denoted as $\widehat{G}^{F^2_\infty}(\Theta)$ with transform variable Θ, as follows (a similar idea of integrating out the initial variance was used by Dragulescu and Yakovenko (2002) to find the density function of log-returns under the Heston model unconditional on the initial value of the variance):

$$\widehat{G}^{F^2_\infty}(\Theta) = \int_0^\infty e^{-(A^F + B^F V')\Theta} G^{v_\infty}(V')dV' = e^{-A^F\Theta}\widehat{G}^{v_\infty}(B^F\Theta), \tag{62}$$

where $\widehat{G}^{v_\infty}(\Theta)$ is specified by (45).

Accordingly, we can compute the probability function of the VIX by:

$$\mathbb{P}[F \leq F'] = \frac{1}{2\pi}\int_0^\infty \Re\left[\frac{e^{F'^2\Theta}}{\Theta}\widehat{G}^{F^2_\infty}(\Theta)\right] d\Theta_I, \ \Theta_R > 0. \tag{63}$$

Based on (63), the daily VIX observations become independent and we can calculate the model implied density and compute model quantiles. As a result, by minimizing the squared difference between the model and empirical quantiles we can estimate model parameters. We estimate model parameters using daily closing values of the VIX for five years of data from 02/28/2003 to 02/29/2008 with total number of observations equal to 1259. Estimated parameters are given in Table 1 under the column titled "Historical". In Figure (5) we show the historical frequency of the VIX and the model implied frequency. In Figure (6) we plot the model quantiles against the empirical data quantiles.

We see that the mean variance jump is quite high compared to the mean variance and about three jumps are expected to occur every two years. These estimates for volatility jump parameters are in line with parameter estimates of volatility jumps obtained in other empirical studies on the S&P 500 index dynamics, which are summarized by Broadie *et al* (2007). From Figures (5) and (6) we also see that the estimated model can adequately describe the right tail of the empirical density of the VIX.

Table 1: Estimated model parameters

	Historical	Implied
$\sqrt{V(0)}$		0.1780
$\sqrt{\theta}$	0.0996	0.1338
κ	2.0553	3.2501
ε	0.0301	0.2897
$\sqrt{\eta}$	0.1492	0.2484
γ	1.4956	1.0727

5.4.2 Implied Calibration

For pricing VIX options consistently with quoted market prices, we calibrate the model parameters by minimizing the squared differences between model prices of VIX options and corresponding market prices, which are computed by averaging bid-ask quotes. We provide an illustration of calibrating the parameters of the model (1) to VIX options data available on July 25, 2007. We calibrate the model to VIX options with maturities one month (Aug) and two (Sep), three (Oct), four (Nov) months (we note that although typically the market is trading in VIX options with six or more maturities only the first three or four are liquid). In Table (1), we report estimated model parameters under the column titled "Implied".

In Figure (7) we show model and market implied volatilities of VIX options which are inferred by applying the Black (1976) formula for the call option on the futures contract to fit either the market quote of the VIX call option or the model price for this option given the quote of the VIX futures contract with the same maturity time. Since a long position in the VIX call option provides a protection against market crashes, when the volatility of the S&P500 index soars, sellers of VIX call options charge corresponding risk premiums, which lead to the positive VIX skew implied from market prices of VIX options.

We see that our model with time-homogeneous parameters fits VIX options implied volatilities very well across different strikes and maturities. By comparing historical and calibrated parameters, we see the market data imply higher variability of the VIX.

6 Approximate Solution for Pricing Problem

Now we consider an approximation of the terminal distribution of the realized variance at time τ by log-normal distribution with appropriately chosen first two moments. We follow the idea employed for approximate pricing of Asian options based on matching two moments of the cumulative spot price to a log-normal random variable (see, for example, Lipton (2001)). We apply this approach to the present context and develop a very accurate approximation of the distribution of $I(\tau)$ for maturities beyond one year, which yields simple analytical formulas for pricing longer-term options on the realized variance. We note that the log-normal approximation to the

Figure 5: Historical and Theoretical Frequency

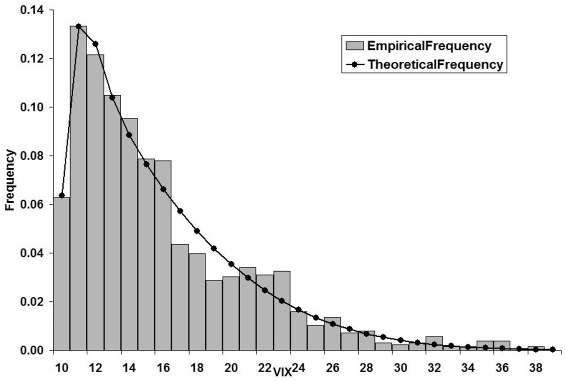
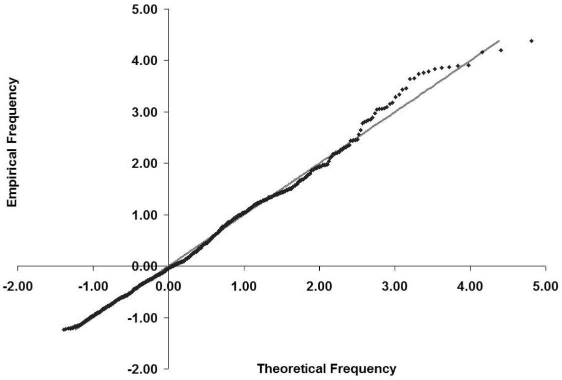


Figure 6: Quantile-Quantile (QQ) Plot



density of the realized variance was also proposed by Friz-Gatheral (2005), however they do not provide any formulas for computing the first and second moments of the realized variance in the Heston model, which is necessary to perform the moment matching.

For brevity we consider the dynamics (1) without jumps in returns and variance. Formally we assume that the realized variance at time τ, $I(\tau)$, can be approximated by log-normal random variable χ:

$$\chi = \alpha \exp\left\{-\frac{\beta^2}{2} + \beta\epsilon\right\}, \tag{64}$$

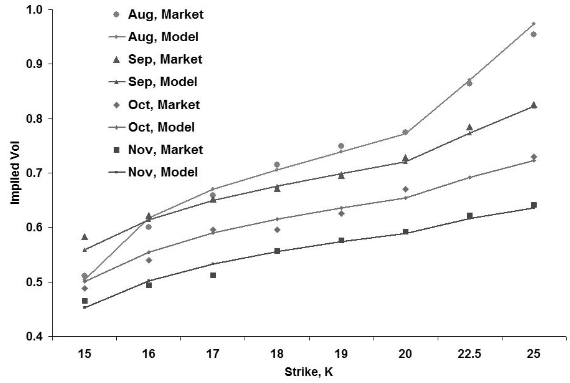
where ϵ is standard normal random variable, $\epsilon \sim N(0,1)$. The first two moments of χ are as follows:

$$\mathbb{M}_1[\chi] = \alpha,\ \mathbb{M}_2[\chi] = \alpha^2 e^{\beta^2}. \tag{65}$$

To find moments of $I(\tau)$ given accumulated realized variance I at time t, $t_0 \leq t < T$, we use the moment generating function of $I(t)$ given by Eq.(20) to calculate:

$$\mathbb{M}_1[I(\tau)] = -\frac{\partial}{\partial \Psi_R}\widehat{G}^I(\tau, V, I, \Psi_R)\mid_{\Psi_R=0},\ \mathbb{M}_2[I(\tau)] = \frac{\partial^2}{\partial \Psi_R^2}\widehat{G}^I(\tau, V, I, \Psi_R)\mid_{\Psi_R=0}. \tag{66}$$

Figure 7: The VIX Volatility Implied by Market and Model Prices



Straightforward calculations yield:

$$
\begin{aligned}
&\overline{I} := \mathbb{M}_1[I(\tau)] = I + \theta\tau + \frac{1}{\kappa}(\theta - V)\left(e^{-\kappa\tau} - 1\right), \\
&\overline{I^2} := \mathbb{M}_2[I(\tau)] = a_2 e^{-2\kappa\tau} + a_1 e^{-\kappa\tau} + a_0, \\
&a_2 = \frac{1}{2\kappa^3}\left(2\kappa(\theta - V)^2 + \varepsilon^2(\theta - 2V)\right), \\
&a_1 = \frac{2}{\kappa^3}\left(\kappa^2(I + \tau\theta)(\theta - V) + \kappa(-(\theta - V)^2 + (\theta - V)\varepsilon^2\tau) + \theta\varepsilon^2\right), \\
&a_0 = (I + \theta\tau)^2 + \frac{1}{2\kappa^3}\left(-4\kappa^2(\theta - V)(I + \tau\theta) + 2\kappa((\theta - V)^2 + \theta\tau\varepsilon^2) + (-5\theta + 2V)\varepsilon^2\right).
\end{aligned}
\tag{67}
$$

Next we match two first moments of $I(\tau)$ and χ to obtain:

$$
\alpha = \overline{I}, \ \beta = \sqrt{\ln \frac{\overline{I^2}}{\overline{I}^2}}. \tag{68}
$$

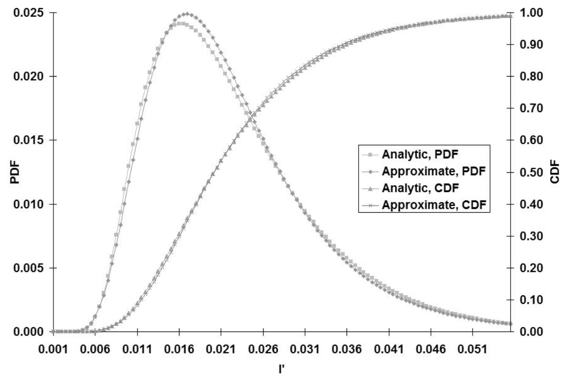
In Figure (8), we show approximation of the true density of $\hat{I}(\tau)$ by log-normal distribution using model parameters from Table (1) under the column titled "Implied" with $\gamma = 0$. It follows that the approximation can adequately fit the true distribution of the realized variance in the Heston model. The approximation becomes especially accurate when time-to-maturity or mean-reversion parameter, κ, increase or if the volatility of the variance, ε, decreases.

Accordingly, we can use χ to derive pricing formulas for options on $I(\tau)$. In particular, for swaps on realized variance we obtain:

$$
\hat{U}^{swp}(\tau, X, V, \hat{I}, \hat{K}, 1) = D(\tau)\left(\frac{1}{(T - t_0)}\alpha - \hat{K}^2\right), \tag{69}
$$

which is in agreement with formula (25) with $\gamma = 0$.

Figure 8: Log-normal Approximation vs True Density of $I(\tau)$ with $\tau = 1.0$



For the value of call option on realized variance swap, we obtain:

$$\frac{\hat{U}^{swco}(\tau, X, V, \hat{I}, \hat{K}, 1)}{D(\tau)} = \frac{\alpha}{(T-t_0)}\mathcal{N}\left(\frac{\ln\left(\frac{\alpha}{(T-t_0)\hat{K}^2}\right) + \frac{\beta^2}{2}}{\beta}\right) - \hat{K}^2\mathcal{N}\left(\frac{\ln\left(\frac{\alpha}{(T-t_0)\hat{K}^2}\right) - \frac{\beta^2}{2}}{\beta}\right), \tag{70}$$

where $\mathcal{N}(x)$ is the standard normal cumulative probability density function.

Similarly we can price other options on the realized variance. In the same way we can use the above approximation for pricing options on the realized volatility. The key observation is that

$$\text{if } \chi = \alpha \exp\left\{-\frac{\beta^2}{2} + \beta\epsilon\right\} \text{ then } \sqrt{\chi} = \widehat{\alpha} \exp\left\{-\frac{\widehat{\beta}^2}{2} + \widehat{\beta}\epsilon\right\}, \tag{71}$$

where $\widehat{\alpha} = \sqrt{\alpha}e^{-\frac{1}{8}\beta^2}$ and $\widehat{\beta} = \frac{1}{2}\beta$. Thus, using these new parameters we can apply formulas (69)-(70) (with normalizing factor $\sqrt{T-t_0}$ along wiht strike $\hat{K}$ and cap $\hat{C}$) for approximate pricing of options on the realized volatility.

7 Hedging

Hedging volatility options is a very important practical issue. Demeterfi *et al* (1999) developed an ingenious dynamic replication strategy for variance swaps involving a strip of European call and put options. However, for more complex options on the realized variance with non-linear payoff functions it is rather difficult to construct such a strategy. Hedging strategies for volatility options and their performance is studied, among others, by Psychoyios-Skiadopoulos (2006), Sepp (2008), Carr-Wu (2008), Broadie-Jain (2008). Here we outline some general principles.

In general, we need to quantify spot-delta, spot-gamma, variance-delta, and variance gamma of hedging positions and underlying instruments. There are two approaches which we can employ for calculating the above mentioned risk sensitivities of volatility options: first, model sensitivities and, second, sticky-strike sensitivities.

To obtain sticky-strike sensitivities we first calibrate the starting set of model parameters corresponding to the current spot price. Then we shift the spot price by one percent up and down and recalibrate the model to get the set of up-parameters and the set of down-parameters. Finally we compute spot-delta and spot-gamma by finite differences. Similarly are computed vega and volga risks. This approach is computationally intensive and might not be accurate depending on the market data and optimization procedure. As a result, we concentrate on model sensitivities.

7.1 Model Sensitivities

To compute model implied sensitivities, we generally differentiate the option value function, which is given by the integral formula (27), and invert the corresponding integral. We note that although the pricing equation does not explicitly depend on the current spot price, the end-of-day realized variance, I_n, does. We note that at the end of the n-th day the accrued variance I_n is computed by (4) given the realized variance at the beginning of the n-th day, I_{n-1}, last fixing price S_{n-1}, and new fixing price S_n.

At this point, we go back to the discrete case and assume that the n-th end-of-day contribution to the realized variance is $\ln^2 \frac{S_n}{S_{n-1}}$ $(n = 1, .., N)$. We denote by $\Delta t = t_n - t_{n-1}$ be time between $(n-1)$-th and n-th fixings (for daily observations and annualization factor $AF = 252$, $\Delta t = \frac{1}{AF}$) and by δ the fraction of the day before the n-th end-of-day fixing, $0 \leq \delta \leq 1$, so that the time left before the next fixing is $\delta \Delta t$. Let S be the current spot price. We then evaluate n-th day's contribution to the end-of-day realized variance, I_n, conditional on the current spot price S, in the following way.

First, for brevity ignoring jumps in returns we assume that the the n-th day closing price is driven by a log-normal process:

$$S_n = S \exp \left\{ \left(r(t_{n-1}) - d(t_{n-1}) - \frac{V(t_{n-1})}{2} \right) \delta \Delta t + \sqrt{V(t_{n-1}) \delta \Delta t} \epsilon \right\}, \tag{72}$$

where $\epsilon \sim N(0,1)$, and $r(t_{n-1})$, $d(t_{n-1})$, $V(t_{n-1})$ are the interest rate, the dividend yield, and the instantaneous variance observed at time t_{n-1}, respectively.

Then we calculate the expected end-of-day realized variance at time t, $t_{n-1} \leq t \leq t_n$, as follows:

$$\widetilde{I} = \mathbb{E}\left[I_n | S\right] = I_{n-1} + \mathbb{E}\left[\ln^2 \frac{S_n}{S_{n-1}} | S\right] = I_{n-1} + \overline{I}(\delta), \tag{73}$$

where the measure is induced by (72) and $\overline{I}(\delta)$ is computed as follows:

$$
\begin{aligned}
\overline{I}(\delta) &= \mathbb{E}\left[\ln^2\frac{S_n}{S_{n-1}}|S\right] = \mathbb{E}\left[\left(\ln\frac{S}{S_{n-1}}+\ln\frac{S_n}{S}\right)^2|S\right] = \\
&= \ln^2\frac{S}{S_{n-1}} + 2\ln\frac{S}{S_{n-1}}\mathbb{E}\left[\ln\frac{S_n}{S}|S\right] + \mathbb{E}\left[\ln^2\frac{S_n}{S}|S\right] = \\
&= \ln^2\frac{S}{S_{n-1}} + 2\ln\frac{S}{S_{n-1}}\left(r(t_{n-1})-d(t_{n-1})-\frac{V(t_{n-1})}{2}\right)\delta\Delta t + V(t_{n-1})\delta\Delta t + O((\delta\Delta t)^2),
\end{aligned}
\tag{74}
$$

In particular, at the beginning of the n-th trading day, $S = S_{n-1}$ and $\delta = 1$, so we obtain:

$$
\widetilde{I} = \mathbb{E}[I_n|S] = I_{n-1} + \overline{I}(1) = I_{n-1} + V(t_{n-1})\Delta t, \tag{75}
$$

which is in agreement with the result we illustrated in expansion (26), where we showed that the leading order contribution, $O(\tau)$, to the expected realized variance is due to the instantaneous variance. Accordingly, our assumption (72) is justified in the short-time limit and we can employ it to calculate (short-term) hedges for volatility options.

We note that we take $I = \widetilde{I}$ rather than $I = I_{n-1}$ in our pricing formulas. Accordingly, we calculate the spot-delta and the spot-gamma for options on the realized variance as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial S}\hat{U}(\tau,X,V,\hat{I},\hat{K},o) = \frac{D(\tau)}{(T-t_0)^o}\frac{\partial}{\partial I}U(\tau,X,V,I,K,o)\frac{\partial}{\partial S}\widetilde{I}, \\
&\frac{\partial^2}{\partial S^2}\hat{U}(\tau,X,V,\hat{I},\hat{K},o) = \frac{D(\tau)}{(T-t_0)^o}\left(\frac{\partial^2}{\partial I^2}U(\tau,X,V,I,K,o)\left(\frac{\partial}{\partial S}\widetilde{I}\right)^2 + \right. \\
&\left. +\frac{\partial}{\partial I}U(t,T,S,V,I,K,o)\frac{\partial^2}{\partial S^2}\widetilde{I}\right),
\end{aligned}
\tag{76}
$$

where

$$
\begin{aligned}
\frac{\partial}{\partial S}\widetilde{I} &= \frac{\partial}{\partial S}\overline{I}(\delta) = \frac{2}{S}\left(\ln\frac{S}{S_{n-1}} + \left(r(t_{n-1})-d(t_{n-1})-\frac{V(t_{n-1})}{2}\right)\delta\Delta t\right), \\
\frac{\partial^2}{\partial S^2}\widetilde{I} &= \frac{\partial^2}{\partial S^2}\overline{I}(\delta) = -\frac{2}{S^2}\left(-1+\ln\frac{S}{S_{n-1}} + \left(r(t_{n-1})-d(t_{n-1})-\frac{V(t_{n-1})}{2}\right)\delta\Delta t\right).
\end{aligned}
\tag{77}
$$

For the variance swap, the hedges computed in this way look similar to those computed from the static decomposition formula by Demeterfi *et al* (1999). In particular, we see that under our proposed method to calculate the model sensitivities, the cash gamma of the variance swap is $\frac{2}{(T-t_0)}$ which agrees with the result reached by Demeterfi *et al* (1999). For options on the realized volatility, we approximate the expected end-of-day realized volatility as follows:

$$
\widetilde{I}^\sigma = \mathbb{E}\left[\sqrt{I_n}|S\right] = \mathbb{E}\left[\sqrt{I_{n-1}+\ln^2\frac{S_n}{S_{n-1}}}|S\right] \approx \sqrt{I_{n-1}+\overline{I}(\delta)}. \tag{78}
$$

Then we are in position to use formulas (76) (with normalizer $\sqrt{(T-t_0)}$), where we take

$$
\begin{aligned}
\frac{\partial}{\partial S}\widetilde{I}^{\sigma} &= \frac{1}{2\sqrt{I_{n-1}+\overline{I}(\delta)}}\frac{\partial}{\partial S}\overline{I}(\delta), \\
\frac{\partial^2}{\partial S^2}\widetilde{I}^{\sigma} &= -\frac{1}{4(I_{n-1}+\overline{I}(\delta))^{3/2}}\left(\frac{\partial}{\partial S}\overline{I}(\delta)\right)^2 + \frac{1}{2\sqrt{I_{n-1}+\overline{I}(\delta)}}\frac{\partial^2}{\partial S^2}\overline{I}(\delta).
\end{aligned}
\tag{79}
$$

7.2 Hedging Jump Risk

Hedging the risk induced by returns and variance jumps is an important practical issue because sudden jumps undermine the effectiveness of the delta-hedging strategy. One way of quantifying the jump risk, initially proposed by Andersen-Andreasen (2000) for equity options, is to evaluate the expected jump impact on the volatility option $U(\tau, V, I, K, o)$:

$$
\overline{U}(\tau, V, I, K, o) = \int_0^{\infty}\int_{-\infty}^{\infty}[U(V+J^v, I+(J^s)^2) - U]\varpi^v(J^v)\varpi^s(J^s)dJ^s dJ^v. \tag{80}
$$

Using the exponential form of the solution (27), we can compute $\overline{U}$ explicitly:

$$
\overline{U}(\tau, V, I, K, o) = \frac{1}{\pi}\int_0^{\infty}\Re\left[\left(\frac{\exp\left\{\frac{-\nu^2\Psi}{2\Psi\delta^2+1}\right\}}{(1-\eta B^I)\sqrt{2\Psi\delta^2+1}} - 1\right)\widehat{G}^I(\tau, V, I, \Psi)\breve{u}(\Psi, K, o)\right]d\Psi_I, \tag{81}
$$

The above result can also be extended in a straitforward way to measure the expected jump impact of vanilla options.

Accordingly, we can evaluate the overall jump impact on a position in different volatility options and hedge it away by taking an offsetting position. A Monte-Carlo simulation given in Sepp (2008) shows that this strategy can potentially eliminate the jump risk associated with a short position in a call option on the VIX. Given the liquidity of VIX options this strategy can also be used to hedge bespoke volatility options by trading in liquid options and futures on the VIX.

8 Numerical Results

For illustration and benchmarking, we present a few calculations using the numerical inversion of our derived formulas. For numerical inversion of our formulas we use Gaussian quadratures and integrate inverse Fourier integrals along the line $\Psi = -1 + i\Psi_I$ with $\Psi_I \in \mathbb{R}^+$. The left integration bound is zero, while the right integration bound is determined locally - when subsequent contributions to the integral become small, the integration is stopped.

For Monte-Carlo pricing, we simulate the market model (1) and sample realized variance according to formula (4) assuming daily fixing and annualization factor equal to $AF = 252$. Accordingly, we set the time step to be $\Delta t = \frac{1}{252}$. We use a simple Euler discretization of the market dynamics (1) with the variance $V(t)$ floored at zero. More refined and efficient schemes can be found in Broadie-Kaya (2006) and Andersen (2008), however since our direct purpose is to provide a benchmark for our analytical methods, we supply Monte Carlo results for a mere verification.

We use model parameters from Table (1) under the column titled "Implied" with three model specifications: 1) SV is the stochastic volatility model without jumps, 2) SVVJ is the SV model with variance jumps only, 3) SVPJ is the SV model with return jumps only with $\nu = -0.1378$ and $\delta = 0.0$. For Monte Carlo simulations we take asset price $S = 1$ and correlation $\rho = -0.5$. The number of path in Monte Carlo is taken to be $10,000$.

We use three annualized maturity times, T, and corresponding number of fixings, N. The strike price, K, is chosen so that the options is about at-the-money. Notional of the contract is taken to be one unit. We use the following notations: NI - option value computed by numerical inversion of Fourier integral, MC - the option value computed by Monte Carlo, $SE(MC)$ - the standard error of Monte Carlo price.

In Tables (2) and (3), we report computed values of variance and volatility swaps, respectively, with zero strike. In Tables (4) and (5), we provide computed values of call options on variance and volatility, respectively. In Table (6), we report values of the forward-start option on the volatility swap with $\tau_F = \tau_T = T$.

We observe some differences between the numerical inversion and Monte Carlo for options with short maturities, although differences for the variance and volatility swaps are not that sizable. The explanation is that the short-term discretely sampled variance has a higher volatility, which increases option values. However, for longer maturities these differences become less pronounced.

Table 2: Swap on Realized Variance

Model	N	T	NI	MC	MC (SE)
SV	20	0.08	3.006389%	3.013725%	0.012297%
SV	126	0.50	2.471996%	2.464380%	0.011889%
SV	252	1.00	2.198141%	2.193963%	0.010388%
SVVJ	20	0.08	3.247888%	3.262310%	0.019990%
SVVJ	126	0.50	3.502018%	3.506299%	0.025598%
SVVJ	252	1.00	3.632280%	3.607559%	0.025313%
SVPJ	20	0.08	5.043377%	5.137316%	0.073016%
SVPJ	126	0.50	4.508919%	4.532617%	0.031122%
SVPJ	252	1.00	4.235074%	4.218653%	0.022276%

9 Conclusions

We have presented a robust analytical approach to handle pricing and hedging options on the realized variance and volatility. We have applied the Heston stochastic

Table 3: Swap on Realized Volatility

Model	N	T	NI	MC	MC (SE)
SV	20	0.08	17.203192%	17.016629%	0.034363%
SV	126	0.50	15.324718%	15.271417%	0.036364%
SV	252	1.00	14.457550%	14.434628%	0.033225%
SVVJ	20	0.08	17.913002%	17.528911%	0.043555%
SVVJ	126	0.50	17.811056%	17.798127%	0.058189%
SVVJ	252	1.00	18.179713%	18.067458%	0.058589%
SVPJ	20	0.08	20.301516%	20.046319%	0.105777%
SVPJ	126	0.50	20.107899%	20.108851%	0.069929%
SVPJ	252	1.00	19.856499%	19.805524%	0.054415%

Table 4: Option on Realized Variance Swap

Model	N	T	K	NI	MC	MC (SE)
SV	20	0.08	0.16	0.561698%	0.706805%	0.009943%
SV	126	0.50	0.16	0.410045%	0.420670%	0.008097%
SV	252	1.00	0.16	0.267108%	0.269308%	0.006347%
SVVJ	20	0.08	0.18	0.417287%	0.601210%	0.016966%
SVVJ	126	0.50	0.18	0.961598%	0.979945%	0.021336%
SVVJ	252	1.00	0.18	1.051085%	1.056063%	0.020849%
SVPJ	20	0.08	0.21	1.938234%	2.110268%	0.068130%
SVPJ	126	0.50	0.21	1.264390%	1.298340%	0.022453%
SVPJ	252	1.00	0.21	0.810298%	0.809281%	0.014270%

volatility model to describe the random evolution of the realized variance, incorporated return and variance jumps to fit market prices of options with short maturities, and obtained closed-form solutions for swaps on the realized variance and volatility as well as for options on these swaps. We have applied our methodology for pricing forward-start volatility options, including options on the VIX. We have developed a new method to estimate the model parameters from time series of the VIX, and by calibrating the model to the historical and market data, we have shown that the model fits both data sets very well. We have derived a log-normal approximation to the density of the realized variance in the Heston model and applied it to obtain approximate solutions for the pricing problem. These approximate solutions are very accurate for maturities beyond one year and they can serve as an alternative tool for pricing volatility options with longer-maturities. We have also considered how to appropriately compute model implied hedges in our framework. Our approach allows to consistently unify pricing and risk-managing of different volatility options into one single framework.

10 Appendix A.1. Derivation of Formula (20)

We solve PIDE (19) by assuming that its solution has an exponential-affine form:

$$\begin{aligned}&\widehat{G}^{I}(\tau,V,I,\Psi)=e^{-\Psi I+A^{I}(\tau,\Psi)+B^{I}(\tau,\Psi)V+\Gamma^{I}(\tau,\Psi)},\\&A^{I}(0,\Psi)=0,\ B^{I}(0,\Psi)=0,\ \Gamma^{I}(0,\Psi)=0.\end{aligned}\tag{82}$$

Table 5: Option on Realized Volatility Swap

Model	N	T	K	NI	MC	MC (SE)
SV	20	0.08	0.16	1.589344%	1.900043%	0.025039%
SV	126	0.50	0.16	1.118588%	1.140349%	0.020678%
SV	252	1.00	0.16	0.735351%	0.739062%	0.016538%
SVVJ	20	0.08	0.18	0.958027%	1.358506%	0.030611%
SVVJ	126	0.50	0.18	2.089152%	2.128883%	0.041017%
SVVJ	252	1.00	0.18	2.326075%	2.319045%	0.040920%
SVPJ	20	0.08	0.21	2.625548%	2.923369%	0.089279%
SVPJ	126	0.50	0.21	2.497760%	2.542702%	0.040715%
SVPJ	252	1.00	0.21	1.675938%	1.672503%	0.028083%

Table 6: Forward-Start Option on Realized Volatility Swap

Model	N	T	K	NI	MC	MC (SE)
SV	20	0.08	0.16	1.612961%	1.793900%	0.029264%
SV	126	0.50	0.16	0.763437%	0.765965%	0.019409%
SV	252	1.00	0.16	0.466409%	0.481200%	0.014287%
SVVJ	20	0.08	0.18	1.848542%	2.054400%	0.046705%
SVVJ	126	0.50	0.18	2.897558%	2.936900%	0.052140%
SVVJ	252	1.00	0.18	2.780100%	2.780500%	0.045394%
SVPJ	20	0.08	0.21	2.765085%	2.926800%	0.087225%
SVPJ	126	0.50	0.21	2.059276%	2.053200%	0.037265%
SVPJ	252	1.00	0.21	1.362380%	1.380800%	0.026073%

Plugging (82) into PIDE (19) gives the following system of ODE-s:

$$\begin{aligned}
A^I_\tau &= \kappa\theta B^I, \\
B^I_\tau &= -\Psi - \kappa B^I + \frac{1}{2}\varepsilon^2 (B^I)^2, \\
\Gamma^I_\tau &= \gamma \int_0^\infty \int_{-\infty}^\infty [e^{B^I J^v - \Psi (J^s)^2} - 1] \varpi^v(J^v) \varpi^s(J^s) dJ^s dJ^v \\
&= \gamma \left(\left(\frac{\exp\left\{\frac{-\nu^2\Psi}{2\Psi\delta^2+1}\right\}}{\sqrt{2\Psi\delta^2+1}} \right) \frac{1}{1-\eta B^I} - 1 \right),
\end{aligned}$$

given $\Psi_R > -1/(2\delta^2)$.

The equation for B^I is solved by introducing function $C(\tau, \Psi)$ such that $C'(\tau) = -\frac{\varepsilon^2}{2} B^I C$ with initial conditions $C(0, \Psi) = 1$ and $C_\tau(0, \Psi) = 0$. Then we get a second order linear ODE for C, by solving which we obtain the solution for B^I ($B^I = \frac{C'}{\frac{\varepsilon^2}{2} C}$) and A^I ($A^I = \frac{-2\kappa\theta}{\varepsilon^2} \ln(C)$). Next we solve for $\Gamma^I(\tau, \Psi)$ by integrating the left-hand side. The final result is given in formula (20).

11 Appendix A.2. Derivation of Formula (40)

We solve PIDE (39) by applying transform in V', $\widehat{G}^v = \mathcal{F}_-[G^V(V')](\Theta)$, which solves PIDE (40) subject to initial condition: $\widehat{G}^v(\tau, V, \Theta) = e^{-V\Theta}$. We assume that the solution to $\widehat{G}^v(\tau, V, \Theta)$ has an exponential-affine form:

$$\begin{aligned}&\widehat{G}^v(\tau, V, \Theta) = e^{A(\tau,\Theta)+B(\tau,\Theta)V+\Gamma(\tau,\Theta)},\\ &A(0,\Theta) = 0,\ B(0,\Theta) = -\Theta,\ \Gamma(0,\Theta) = 0,\end{aligned} \tag{83}$$

and then we follow the same steps as in Appendix A1 to obtain the final solution given by formula (40).

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