SWAP RATE VARIANCE SWAPS

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Abstract (first posting at SSRN, June 30th, 2008)

We study the hedging and valuation of variance swaps defined on a swap interest rate. Our motivation is the recognition of the fundamental role of variance swaps in the transfer of variance risk, and the extensive empirical evidence documenting that the variance realized by interest rates is stochastic. Working in a diffusion setting, we identify a replication rule as the difference between a static European contract and the gains of a dynamic position on interest rate swaps. Two distinguishing features arise in the context of interest rates: the nonlinear and multidimensional relationship between the values of the dynamically traded contracts and the underlying rate, and the possible stochasticity of the interest rate used for reinvesting dynamic gains. The combination of these two features leads to additional terms in the cumulative dynamic trading gains, terms which depend on realized variance and are taken into consideration in the determination of the appropriate static hedge. We characterize the static payoff function as the solution of an ordinary differential equation, and derive explicitly the associated dynamic strategy. We use daily interest rate data between 1997 and 2007 to test the effectiveness of our hedging methodology and verify that the replication error is small compared to the bid-ask spread in swaption prices.
1 Introduction

An extensive empirical literature has documented that the variance realized by interest rates is stochastic. Stochastic volatility as in Andersen and Lund (1997) and Ball and Tourous (2000), random jumps as in Johannes (2004), switching regimes as in Gray (1996), are just a few examples of the statistical regularities observed in the behavior of interest rates that lead to stochastic realized variance. An investor that recognizes uncertainty about future realized variance as a risk factor distinct from the risk associated to the level of rates might want to modify his exposure with respect to variance risk. This is possible by trading in a simple contract that depends exclusively on variance risk: the variance swap. In this contract, two parties agree at $t = 0$ to exchange a dollar amount proportional to the variance realized by a reference interest rate between 0 and $T$, against a fixed sum set at inception. The valuation problem associated to a variance swap is the computation of the fixed payment that makes the contract worthless at inception.

In the absence of other liquidly traded instruments that could be used as hedges, the valuation of a variance swap must take into account subjective probabilities for outcomes in realized variance as well as preferences toward variance risk. However, it was shown independently by Neuberger (1992) and Dupire (1993) that the variance realized by a traded asset following a diffusion can be replicated as the difference between a static European contract on the asset and the cumulative gains that arise from a dynamic position in the same asset. Moreover, by Breeden and Litzenberger (1978) and Carr and Madan (1998) (see also Demeterfi et al., 1999), the payoff of the static contract can be replicated by a sufficiently rich portfolio of call and put European options. Strikingly, the replicating portfolio is derived under very mild assumptions about the volatility process of the traded asset. Therefore, the arbitrage-free price of realized variance is determined by the price of market observable European call and put options.

In reality, the existence of a rich set of liquidly traded options creates an alternative motivation for trading variance swaps. Future variance can be understood as a fundamental quantity simultaneously driving the prices of options and variance swaps, therefore a variance swap can
be used to set a speculative position by an investor who believes that option prices are far from
his own subjective valuation. A variance swap provides the investor with a tool to speculate
on future variance, \textit{relative to the variance implied in observable option prices}, without taking
any risk related to the direction of changes in the underlying.

Following the successful development of variance swaps defined on individual stocks and on
equity indices, the literature has moved toward increasingly sophisticated computational ap-
proaches and more complex payoff structures. Little and Pant (2001) studied finite difference
methods for variance swaps with discrete sampling in a local volatility model. Carr and Lewis
(2004) explored the valuation and hedging of corridor variance swaps. The universe of con-
tract payoffs has been expanded to include volatility swaps and volatility derivatives. Howison,
Rafailidis and Rasmussen (2004) priced volatility derivatives in a partial differential equation
framework with emphasis on an asymptotic analysis for fast mean reverting volatility. Java-
heri, Wilmott and Haug (2004) studied volatility swaps in a GARCH setting and derived an
approximate solution for the convexity correction between variance and volatility. Carr et al.
(2005) priced options on realized variance under a pure jump process. Windcliff et al. (2006)
focused on the hedging of discretely sampled volatility derivatives and the effects of jumps and
hedging frequency. Carr and Lee (2008a) developed a robust and nonparametric replication
methodology for volatility derivatives under an independence assumption between the under-
lying and its volatility. Broadie and Jain (2008) studied the pricing and risk management of
volatility derivatives under Heston stochastic volatility model. The variance swap has also been
used by Carr and Wu (2008) as a tool to extract market information about variance risk premia
from observed option prices. Finally, in work done mutually independently from this paper,
Carr and Lee (2008b) studied the hedging of variance options and obtained a replication result
for the weighted variance of a general function of a positive, continuous semimartingale price
process. As in our paper, they characterize the static hedging payoff as a solution of an ordinary
differential equation.

In this paper we investigate variance swaps in the context of interest rates, previously
unexplored in the literature. We focus on the explicit valuation and hedging of variance swaps
defined on the volatility of interest rates derived from the Libor curve. This is the yield curve used in the dominant over-the-counter interest rate derivatives market. Brigo and Mercurio (2006) thoroughly cover commonly traded interest rate derivatives and related pricing models. As in earlier work in variance swaps, we identify a replication strategy formulated in terms of a static European contract and a dynamic trading rule. We design a dynamic strategy that is consistent with current market practice by using the most flexible and liquidly traded at-the-money forward interest rate swaps and Libor deposits.

Two issues distinguish the hedging and valuation problem in the context of interest rates: First, the relationship between the underlying over which variance is computed (forward swap rate), and the values of available traded instruments (swaps, bonds, Libor deposits) is nonlinear and multidimensional. A forward swap rate is not a traded security. Second, the contract depends on the variance of interest rates. Therefore, assuming deterministic interest rates for reinvesting dynamically accrued gains, as is standard in the literature on variance swaps, is suspect. Instead, we choose to reinvest gains using Libor deposits at the prevailing (possibly stochastic) interest rate. We combine these two features under the assumption of a sufficiently flat forward curve, which allows us to transform the high dimensional hedging problem into a one dimensional computation, depending only on the dynamics of the swap rate over which variance is recorded. We obtain a closed form representation for the cumulative gains of a dynamic trading strategy including novel terms that depend on realized variance. This requires the identification of an appropriate static payoff function and associated dynamic strategy for the difference between them to be precisely the variance we intend to replicate. We characterize the static payoff function as the solution of an ordinary differential equation, compute its solution numerically, and derive from it a dynamic trading strategy. We perform this exercise for the replication of the variance of increments in the underlying swap rate as well as for the variance of returns. In absence of arbitrage, the initial value of the fixed leg of the variance swap must coincide with the initial value of the replicating portfolio that delivers realized variance. This includes an initially costless dynamic position, therefore the valuation of the variance swap is strictly equivalent to the valuation of the European static payoff. The effectiveness of
the hedging strategy is tested empirically by comparing historically realized variance with the performance of the hedging portfolio, using historical interest rate data with daily frequency between 1997 and 2007. This, implicitly, is also a test of appropriateness of the diffusive dynamics adopted in this paper. We find that the hedging methodology has small errors relative to the bid-ask spread of traded swaptions.

The paper is structured as follows. Section 2 describes the features of the variance swap contract. Section 3 reviews the valuation of a variance swap contract defined over a traded asset under reinvestment with zero rates. Section 4 presents the forward rate model, its connection with swap rates, and an approximation that allows us to reduce the high dimensionality of the yield curve to a single underlying. In Section 5 we present the main theoretical result of the paper: an explicit hedging strategy based on a characterization of the static European contract and a dynamic position in swaps. Section 6 contains empirical tests of the hedging methodology. Our conclusions are in Section 7.

2 The variance swap contract

A variance swap is a derivative security defined on an underlying financial variable $S$. We model the underlying dynamics with a continuous time stochastic process $S_t$, with time measured in years. The variance swap is a contract between two parties who agree to exchange, at time $T$, the difference between the empirical variance realized by $S_t$ on $[0, T]$, and a fixed amount $T\beta^2$ agreed upon at $t = 0$. We assume that there are 252 trading days per year. For an integer $N > 0$ we consider two types of contracts: an arithmetic variance swap contract, with payoff at $T = N/252$

$$V_T = \sum_{i=1}^{N} (S_{\frac{i}{252}} - S_{\frac{i-1}{252}})^2 - T(\beta_n)^2,$$  

(1)

and a geometric variance contract, with payoff at $T$

$$V_T = \sum_{i=1}^{N} \left( \frac{S_{\frac{i}{252}} - S_{\frac{i-1}{252}}}{S_{\frac{i-1}{252}}} \right)^2 - T(\beta_l)^2.$$  

(2)
For example, $S$ could be a traded asset, as in Section 3 in this paper, or an interest rate, as in Section 5.

In our analytical work we rely on continuous time processes to derive hedging formulas for continuously sampled variance. This implies an approximation, which is standard in much of the variance swap literature, as the variance delivered by real contracts is recorded by squaring increments over discrete time ((1) and (2)). However, in the empirical tests in Section 6 we take the dynamic rule derived for continuous execution and implement it in discrete time, with the same sampling frequency at which increments for variance are recorded.

3 Valuation in the case of a traded underlying

In order to illustrate the essence of the valuation methodology we review first the case in which the variance swap is defined over a traded asset and gains accumulated dynamically are reinvested at zero interest rate (the case of nonzero, deterministic interest rates can be solved essentially along the same lines). We consider the arithmetic variance case, because of its importance in the interest rate market, and the geometric variance case, previously covered by Dupire (1993), Carr and Madan (1998), and Demeterfi et al. (1999). We assume a filtered probability space $(\Omega, F, F_t, P)$ under the usual hypotheses. The traded asset, under the physical measure $P$, follows

$$dS_t = \mu_t dt + \gamma^n_t dW_t,$$

for $\mu_t, \gamma_t$ adapted to the filtration $F_t$ and $W_t$ a $P$-Brownian motion. We assume that $\mu_t, \gamma^n_t$ are such that $0 < S_t$. Applying Ito’s lemma on $(S_t - S_0)^2$ leads to

$$d(S_t - S_0)^2 = 2(S_t - S_0)dS_t + (\gamma^n_t)^2 dt.$$  

Integrating we get

$$(S_T - S_0)^2 - 2 \int_0^T (S_t - S_0)dS_t = \int_0^T (\gamma^n_t)^2 dt. \quad (3)$$
The continuous time arithmetic variance of $S$ realized between 0 and $T$, \( \int_0^T (\gamma_t^n)^2 dt \), is represented in (3) as the terminal gain of a portfolio composed by two pieces: a static position on a quadratic contract with final payoff \((S_T - S_0)^2\), and the gains of a dynamic position in $S$, with notional $-2(S_t - S_0)$. Remarkably, (3) has been derived under very mild assumptions about the volatility process $\gamma_t^n$ which, in particular, could be stochastic.

The initial value of future realized variance must be equal to the initial value of the replicating portfolio. In order to preclude arbitrage, we assume the existence of a risk neutral probability measure $Q$ that makes discounted assets martingales. Because $S_t$ is an asset price and we have assumed zero interest rates, it follows that $S_t$ itself must be a martingale. Therefore, by Girsanov’s theorem we have

\[
\frac{dS_t}{S_t} = \gamma_t^n dW^Q_t,
\]

with $W^Q_t$ a Brownian motion under the risk neutral measure $Q$. The initial value of the dynamic component of the replication strategy is

\[
-2E^Q[\int_0^T S_t dS_t | F_0] = 0,
\]

by virtue of the martingale condition in (4). The static term in the hedging portfolio is a quadratic European payoff and can be priced explicitly by a replicating portfolio composed of European call and put options, as shown in Carr and Madan (1998). Therefore, the arbitrage-free value of future realized variance is determined by the price of liquidly traded European options. For simplicity, in this paper we will focus on the decomposition of realized swap rate variance up to a European payoff and a dynamic traded position, and will not be concerned with the decomposition of the European payoff into calls and puts.

A similar argument to that outlined above can be followed to identify a hedging portfolio that replicates realized geometric variance. In this case we postulate that the asset dynamics under the physical measure is
\[ dS_t = S_t \mu_t dt + S_t \gamma_t \gamma_t dW_t, \]

with \( \mu_t \) and \( \gamma_t \) adapted to the filtration \( F_t \), apply Ito’s lemma on \( \ln(S_t/S_0) \) and integrate between 0 and \( T \) to get

\[
\ln(S_T/S_0) = \int_0^T \frac{1}{S_t} dS_t - \frac{1}{2} \int_0^T \frac{1}{S_t^2} S_t^2 (\gamma_t)^2 dt.
\]

Therefore

\[
-2 \ln(S_T/S_0) + 2 \int_0^T \frac{1}{S_t} dS_t = \int_0^T (\gamma_t)^2 dt. \tag{6}
\]

The realized geometric variance to be delivered at \( T \) is replicated in (6) in terms of a static logarithmic contract and a dynamic position on the asset to be rebalanced between 0 and \( T \).

A common feature of the two examples shown in this section is that the variance of interest is replicated through the gains of a suitable static payoff minus the gains accrued by a dynamic position in the underlying. We will see that this structure is preserved in the replication of swap rate variance, although for a more complicated static payoff function and dynamic trading rule.

4 Forward curve model

Swap rates are the most widely quoted interest rates for derivative transactions between high credit quality banks. Swap rate levels and their implied volatilities are readily extracted from very liquidly traded instruments. This is the motivation for defining contracts on the variance of a forward swap rate, as investigated in this paper. However, unlike in the examples explored in the previous section, swap rates are not traded assets. The most liquidly traded contracts defined on long dated Libor curve rates are swaps. And the value of a swap depends not only on the underlying swap rate, but on the yield curve up to the maturity of the swap.

As we intend to hedge variance contracts by trading in interest rate swaps, we must begin by specifying a general model of forward rates for a consistent description of swap rate dynamics and swap valuation.
Following Heath, Jarrow, and Morton (1994) we consider a family of instantaneous, continuously compounded forward rates \( f_t(u) \) for \( 0 \leq t \leq u, u \leq T^* \). The value of a zero coupon bond expiring at \( T \leq T^* \), for \( t \leq T \), is denoted by \( B_t(T) \). Bond prices and forward rates are related through:

\[
B_t(T) \equiv e^{-\int_t^T f_t(u) du}.
\]  

(7)

Investing 1 dollar at \( t \) in a Libor deposit that expires at \( T \) is equivalent to buying \( 1/B_t(T) \) units of the zero coupon bond expiring at \( T \) and holding it until expiration.

Swap rates and related derivatives are associated to a discrete tenor structure—a finite set of dates with arbitrary start \( T_0, 0 \leq T_0 < T_1 < \cdots < T_M < T^* \), with \( T_{i+1} - T_i \equiv \delta \). The fixed accrual period \( \delta \) is expressed as a fraction of a year; for instance, \( \delta = 1/2 \) represents six months.

Discrete forward Libor rates \( L(T_0), \ldots, L(T_{M-1}) \) associated to the tenor structure are defined from bond prices (7) as in Musiela and Rutkowski (1997) by setting

\[
L_t(T_k) = \frac{1}{\delta} \left( \frac{B_t(T_k)}{B_t(T_{k+1})} - 1 \right) = \frac{1}{\delta} \left( e^{\int_{T_k}^{T_{k+1}} f_t(u) du} - 1 \right), \quad t \in [0, T_k], \quad k = 0, \ldots, M - 1.
\]  

(8)

Forward swaps are widely traded derivatives. A payer’s swap holder makes fixed payments \( \delta K \) and receives floating payments \( \delta L_{T_i}(T_i) \) at \( T_{i+1} \), \( i = 0, \ldots, M - 1 \). The swap value at the beginning of the setting of the first floating payment is

\[
V_{T_0}(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_{T_0}(T_{j+1})(S_{T_0}(T_0, T_M) - K),
\]

where the swap rate \( S(T_0, T_M) \) at \( t \) is

\[
S_t(T_0, T_M) = \sum_{j=0}^{M-1} b_t(j) L_t(T_j) \quad \text{with} \quad b_t(j) \equiv \frac{B_t(T_{j+1})}{\sum_{i=0}^{M-1} B_t(T_{i+1})}.
\]  

(9)

The forward swap price at \( t < T_0 \) is computed under the pricing measure \( P^{0,M} \) that uses

\[
D_t(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_t(T_{j+1}),
\]  

(10)
as numeraire, see Brigo and Mercurio (2006) for a detailed treatment. In this case we get

\[ V_t(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_t(T_{j+1}) E^{P_{0,M}} [(S_{T_0}(T_0, T_M) - K) | F_t]. \] (11)

Furthermore, from the representation

\[ S_t(T_0, T_M) = \frac{B_t(T_0) - B_t(T_M)}{\delta \sum_{j=0}^{M-1} B_t(T_{j+1})}, \] (12)

and the form of the numeraire in (10), it is evident that \( S_t(T_0, T_M) \) is a discounted asset price, therefore a martingale under \( P_{0,M} \). Therefore (11) becomes

\[ V_t(T_0, T_M) = \delta \sum_{j=0}^{M-1} B_t(T_{j+1}) (S_t(T_0, T_M) - K) = D_t(T_0, T_M)(S_t(T_0, T_M) - k). \] (13)

### 4.1 Flat forward curve approximation

As presented so far, the shape and level of the forward curve is unconstrained. In this section we introduce a set of assumptions about the forward curve that facilitate the development of an analytical treatment of the variance swap.

**Assumption 4.1** For \( t \leq T_0 \leq u \leq T_M \), there exists \( \epsilon << 1 \) such that

(i) \( 0 < f_t(u) < \epsilon \),

(ii) \( \max_{u \in [T_0, T_M]} f_t(u) - \min_{u \in [T_0, T_M]} f_t(u) < \epsilon^2. \)

Assumption 4.1 imposes interest rates much lower than 100 % per annum and a relatively flat forward curve. It is important to notice that Assumption (4.1) is static, and is postulated to hold at all times. We are not concerned here with the identification of the class of dynamic models for which the evolution of the yield curve satisfies Assumption (4.1).

From Assumption 4.1 and (8) it follows that for fixed \( t \) and \( j = 0, ..., M - 1, \)
\[ \frac{1}{\delta}(e^{\delta \min_{u \in [T_0, T_M]} f_t(u)} - 1) \leq L_t(j) \leq \frac{1}{\delta}(e^{\delta \max_{u \in [T_0, T_M]} f_t(u)} - 1). \]  

(14)

Since \( \delta < 1 \), and \( 0 < \min_{u \in [T_0, T_M]} f_t(u) \leq \max_{u \in [T_0, T_M]} f_t(u) < \epsilon \), it follows from an exact truncated Taylor expansion that

\[ e^{\delta \min_{u \in [T_0, T_M]} f_t(u)} = 1 + \delta \min_{u \in [T_0, T_M]} f_t(u) + 1/2K_1(\delta \min_{u \in [T_0, T_M]} f_t(u))^2, \]

and

\[ e^{\delta \max_{u \in [T_0, T_M]} f_t(u)} = 1 + \delta \max_{u \in [T_0, T_M]} f_t(u) + 1/2K_2(\delta \max_{u \in [T_0, T_M]} f_t(u))^2, \]

with unknown constants \( K_1, K_2 \in [0, 1] \). Therefore

\[ \min_{u \in [T_0, T_M]} f_t(u) \leq L_t(j) \leq \max_{u \in [T_0, T_M]} f_t(u) + 1/2\delta(\max_{u \in [T_0, T_M]} f_t(u))^2. \]  

(15)

The forward swap rate \( S_t(T_0, T_M) \) can be interpreted through (9) as a weighted average of forward Libor rates, \( L_t(T_0), ..., L_t(T_{M-1}) \). Moreover, because bond prices are positive, weights defined as

\[ \frac{B_t(T_{j+1})}{\sum_{i=0}^{M-1} B_t(T_{i+1})}, \quad j = 0, ..., M - 1, \]

are also positive and add up to one. It then follows immediately from (15) that

\[ \min_{u \in [T_0, T_M]} f_t(u) \leq S_t(T_0, T_M) \leq \max_{u \in [T_0, T_M]} f_t(u) + 1/2\delta(\max_{u \in [T_0, T_M]} f_t(u))^2. \]  

(16)

4.2 Bond ratio approximation

Fixing \( \delta = 0.5 \) at its standard value in the interest rate derivatives market, we use bound (16) to state an approximate relationship between the forward swap rate \( S_t(T_0, T_M) \) and the ratio of bond prices \( \frac{B_t(T_M)}{B_t(T_0)} \).
PROPOSITION 4.1 If Assumption 4.1 holds, then the yield implied by a ratio of bonds,

\[ y_t(T_0, T_M) \equiv -\frac{1}{(T_M - T_0)} \ln\left( \frac{B_t(T_M)}{B_t(T_0)} \right) \]

satisfies

\[ |y_t(T_0, T_M) - S_t(T_0, T_M)| \leq \frac{5}{4} \epsilon^2. \]

Proof:

\[ \left| \frac{-1}{(T_M - T_0)} \ln\left( \frac{B_t(T_M)}{B_t(T_0)} \right) - S_t(T_0, T_M) \right| = \left| \frac{1}{(T_M - T_0)} \int_{T_0}^{T_M} f_t(u) du - S_t(T_0, T_M) \right|. \]

Invoking (16) we write

\[ \left| \frac{1}{(T_M - T_0)} \int_{T_0}^{T_M} f_t(u) du - S_t(T_0, T_M) \right| \leq \max \left\{ \left| \max_{u \in [T_0, T_M]} f_t(u) - \min_{u \in [T_0, T_M]} f_t(u) \right|, \right. \]

\[ \left. \left| \min_{u \in [T_0, T_M]} f_t(u) - \left( \max_{u \in [T_0, T_M]} f_t(u) + 1/2\delta (\max_{u \in [T_0, T_M]} f_t(u))^2 \right) \right| \right\} \leq \frac{5}{4} \epsilon^2, \]

where the last inequality follows from Assumption 4.1.

It is in the sense of Proposition (4.1) that we write

\[ \frac{B_t(T_M)}{B_t(T_0)} \approx e^{-(T_M - T_0)S_t(T_0, T_M)}. \] (17)

5 Hedging and valuation

5.1 Preliminaries

In this section we derive a replicating portfolio that delivers swap rate realized variance at
terminal time \( T_0 \). As in Section 3, the essence of the method consists on replicating realized
variance as the difference between a suitable static payoff and the gains that arise from a
dynamic position. We depart from Section 3 in the fact that the underlying swap rate is not
directly tradeable, and by reinvesting dynamic gains at the, possibly stochastic, rate implied by $B_t(T_0)$.

We consider a variance swap defined on the variance realized between 0 and $T_0$ by the swap rate corresponding to an interest rate swap defined on a discrete tenor structure as in Section 4, beginning at $T_0$ and ending at $T_M < T^*$. The starting and ending dates $T_0$ and $T_M$, associated with the swap rate, are fixed for all $t \leq T_0$. We write $S_t$ for $S_t(T_0, T_M)$. Market participants are typically interested in realized variance for relatively short term horizons, defined on swap rates of commonly quoted length. For example, to fix ideas, we might take $T_0 = 0.5$ (6 months) and $L = 10$ (10 years).

We assume a filtered probability space $(\Omega, F, F_t, P)$ under the usual hypotheses, and postulate that the forward swap rate $S_t$, under the physical measure $P$, follows

$$dS_t = \mu_t dt + \gamma^n_t dW_t,$$

where $W_t$ is a $P$–Brownian motion. The processes $\mu_t$ and $\gamma^n_t$ are progressively measurable with respect to $F_t$. We assume square integrability of the volatility process:

$$E[\int_0^{T^*} (\gamma^n_t)^2 dt] < \infty.$$

We also assume that $\mu_t$ and $\gamma^n_t$ are such that $S_t > 0$ for $t \leq T^*$. We define the geometric instantaneous volatility as

$$\gamma_t^l \equiv \frac{\gamma^n_t}{S_t}.$$

By Proposition B.1.2 in Musiela and Rutkowski (1997), $S_t$ is a square integrable semimartingale.

### 5.2 Dynamic swap gains

Since our hedging strategy involves trading in forward interest rate swaps, we begin by relating the gains of a dynamic strategy in swaps to the evolution of the underlying swap rate. Denote $V^*_t$ for the value at $t \leq T_0$ of a forward interest rate swap starting at $T_0$ and ending at $T_M$ with
strike \( S_r \) set at \( \tau \leq t \). In the expression for the swap value (13) the annuity factor (10) can be written as a ratio of bond prices and the swap rate using (12)

\[
V_t^\tau = \frac{B_t(T_0) - B_t(T_M)}{S_t} (S_t - S_r).
\] (21)

By definition, \( V_t^\tau = 0 \).

Our valuation algorithm relies on a replication argument at expiration \( T_0 \) involving a dynamic strategy executed between 0 and \( T_0 \), in which local swap gains are reinvested through Libor deposits up to \( T_0 \). The dynamic strategy is the limit, as the time between portfolio rebalancings tends to zero, of the discrete trading rule described next.

Let \( h : \mathbb{R} \rightarrow \mathbb{R} \) be deterministic and smooth. We write \( h_t \) for \( h(S_t) \). For \( i = 0, \ldots, n - 1 \) consider a portfolio holding, between \( t = iT_0/n \) and \( t = (i+1)T_0/n \), \( h_{iT_0/n} \) units of the forward swap defined on the swap rate \( S_t \) with strike \( S_{iT_0/n} \). Gains are collected at \( (i+1)T_0/n \) and reinvested in a Libor deposit of maturity \( T_0 \). Let \( G(h, n) \) be the value at \( T_0 \) of gains accumulated by following this trading strategy for \( i = 0, \ldots, n - 1 \). We have

\[
G(h, n) \equiv \sum_{i=0}^{n-1} h_{iT_0/n} (V_{iT_0/n}^{iT_0/n} - V_{(i+1)T_0/n}^{iT_0/n})/B_{(i+1)T_0/n}(T_0),
\] (22)

therefore recalling that \( V_{iT_0/n}^{iT_0/n} = 0 \), and using (21), we get

\[
G(h, n) \equiv \sum_{i=0}^{n-1} h_{iT_0/n} 1 - e^{-S_{(i+1)T_0/n}^M/T_0} (S_{(i+1)T_0/n}^M - T_0) (S_{(i+1)T_0/n} - S_{iT_0/n}).
\] (23)

Under Assumption 4.1, the ratios of bond prices in (23) can be replaced by (17) to define the approximate dynamic gain \( \hat{G}(h, n) \)

\[
\hat{G}(h, n) \equiv \sum_{i=0}^{n-1} h_{iT_0/n} 1 - e^{-S_{(i+1)T_0/n}^M/T_0} (S_{(i+1)T_0/n}^M - T_0) (S_{(i+1)T_0/n} - S_{iT_0/n}).
\] (24)

The following proposition provides a representation of the value of \( \hat{G}(h, n) \) as the interval between consecutive portfolio updates tends to zero \( (n \rightarrow \infty) \).
PROPOSITION 5.1 As \( n \to \infty \), \( \hat{G}(h, n) \) converges in probability to
\[
\int_0^{T_0} h_t \frac{1 - e^{-S_t(T_M - T_0)}}{S_t} \quad dS + \int_0^{T_0} h_t \frac{-1 + (1 + S_t(T_M - T_0))e^{-S_t(T_M - T_0)}}{S_t^2} \quad \gamma_n^2 dt
\]

We write \( \hat{G}(h) \equiv \lim_{n \to \infty} \hat{G}(h, n) \).

Proof:

Lighten the notation by introducing \( \epsilon_i \equiv (S_{(i+1)\frac{T_0}{n}} - S_{i\frac{T_0}{n}}) \) for the increment in the underlying process. We write (24) as
\[
\hat{G}(h, n) = \sum_{i=0}^{n-1} h_{\frac{i\tau_0}{n}} 1 - e^{-\frac{-(S_{i\frac{T_0}{n}} + \epsilon_i)(T_M - T_0)}{S_{i\frac{T_0}{n}} + \epsilon_i}} \quad \epsilon_i.
\]

Because \( h \) has been assumed smooth, the coefficient multiplying \( \epsilon_i \) is also smooth in \( S_{\frac{i\tau_0}{n}} \) and \( \epsilon_i \). A Taylor expansion for small \( \epsilon \) leads to
\[
\hat{G}(h, n) = \sum_{i=0}^{n-1} h_{\frac{i\tau_0}{n}} 1 - e^{-\frac{S_{\frac{i\tau_0}{n}}(T_M - T_0)}{S_{\frac{i\tau_0}{n}} + \epsilon_i}} \quad \epsilon_i^2
\]
where \( R(\epsilon_i) \) is the exact remainder of the Taylor expansion containing terms of order 3 and higher in \( \epsilon_i \). Next, take the limit \( n \to \infty \). By Theorem 2.21 in Medvegyev (2007), the sum of terms linear in \( \epsilon_i \) is convergent in probability to
\[
\int_0^{T_0} h_t \frac{1 - e^{-S_t(T_M - T_0)}}{S_t} \quad dS,
\]
because the integrand is adapted and regular. The sum of terms of order 2 in \( \epsilon_i \) converges in probability to
\[
\int_0^{T_0} h_t \frac{-1 + (1 + S_t(T_M - T_0))e^{-S_t(T_M - T_0)}}{S_t^2} \quad \gamma_n^2 dt
\]
by finite quadratic variation of $S_t$. And the sum for higher order terms tends to zero in probability as higher order variations vanish. (Problem 5.11 in chapter 1 of Karatzas and Shreve (1991)).

We have characterized the gains that arise from dynamically trading in swaps with notional $h(S_t)$, for deterministic $h$, and continuous reinvestment through Libor deposits. Our next step is to use Proposition 5.1 to identify an appropriate static payoff hedge to isolate realized swap rate variance.

### 5.3 The static hedge and its differential equation

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, to be interpreted as the payoff of a European contract when acting on $S_{T_0}$. We write $f_t$ for $f(S_t), t \leq T_0$. Applying Ito’s rule on $f_t$ and integrating on $[0, T_0]$ we have

$$f_{T_0} - f_0 - \int_0^{T_0} f'_t dS_t = \int_0^{T_0} \frac{1}{2} f''_t(\gamma^n_t)^2 dt.$$  

(25)

In order to express the integral with respect to $S_t$ in (25) as the accumulated gain of a dynamically traded strategy in swaps and Libor deposits plus some additional variance related terms, we take

$$f'_t = h_t \frac{1 - e^{-S_t(T_M-T_0)}}{S_t},$$

which implies

$$h_t = \frac{f'_t S_t}{1 - e^{-S_t(T_M-T_0)}},$$

and invoke Proposition 5.1 to obtain

$$f_{T_0} - f_0 - \hat{G}'\left(\frac{f'_t S_t}{1 - e^{-S_t(T_M-T_0)}}\right) = \int_0^{T_0} \frac{1}{2} f''_t - \frac{f'_t S_t}{1 - e^{-S_t(T_M-T_0)}} \left(1 + (1 + S_t(T_M - T_0))e^{-S_t(T_M-T_0)} \right) (\gamma^n_t)^2 dt,$$

(26)

which simplifies to
\[ f_{T_0} - f_0 - \dot{G} \left( \frac{f'_t S_t}{1 - e^{-S_t(T_M - T_0)}} \right) = \int_0^{T_0} \left( \frac{1}{2} f''_t + f'_t \left( \frac{1}{S_t} - \frac{(T_M - T_0)e^{-(T_M - T_0)S_t}}{1 - e^{-(T_M - T_0)S_t}} \right) \right) (\gamma_t)^2 dt. \] (27)

Notional \( h_t = \frac{f'_t S_t}{1 - e^{-S_t(T_M - T_0)}} \) is more easily interpreted recalling (17) and (21) to recognize that \( h_t \approx \frac{f'_t B_t(T_0)}{D_t} \). The bond price in the numerator of the notional function arises from the fact that gains are reinvested as a Libor deposit, and the denominator is a consequence of the fact that the position involves swaps.

The integral on the right side of (27) is a (path dependent) weighted sum of realized variance. Therefore, for a trading strategy that replicates arithmetic variance we must identify \( f \) such that the weighting function is constant over time. Let \( L = T_M - T_0 \). We look for \( f = f(x) \) that satisfies

\[ \frac{1}{2} f'' + f' \left( \frac{1}{x} - \frac{Le^{-xL}}{1 - e^{-xL}} \right) = 1. \] (28)

And (20) implies that to replicate geometric variance we need to find \( f = f(x) \) that solves

\[ \frac{1}{2} f'' + f' \left( \frac{1}{x} - \frac{Le^{-xL}}{1 - e^{-xL}} \right) = \frac{1}{x^2}. \] (29)

Notice that terms proportional to \( f \) are absent in (28) and (29). Therefore, each of these equations can be transformed into a system of first order differential equations. For (28) we define \( g \equiv f' \) and write

\[ \frac{1}{2} g' + g \left( \frac{1}{x} - \frac{Le^{-xL}}{1 - e^{-xL}} \right) = 1, \]
\[ f' = g. \] (30)

We solve the system (30) sequentially, computing first \( g \) and then using it as input in the computation of \( f \). Boundary conditions for (28) are chosen to be

\[ f(S_0) = f'(S_0) = 0. \]
With these boundary conditions, the solution to (28) agrees at \( x = S_0 \), in level and slope, with the payoff function used in Section 3 for the variance in the traded underlying case \( f(x) = (x - S_0)^2 \). The solution to (28) is computed virtually instantly with a standard numerical integrator and shown in Figure 1 for \( S_0 = 0.06 \) and \( L = 10 \) (10 years). Figure 2 shows how this solution differs from the traded asset case payoff function \( f(x) = (x - S_0)^2 \).

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Figure 2, now at the end of the document, should be inserted here.

For (29) we define \( g \equiv f' \) and write

\[
\frac{1}{2} g' + g \left( \frac{L e^{-xL}}{1 - e^{-xL}} \right) = \frac{1}{x^2},
\]

that we solve numerically as in the arithmetic variance case, with the boundary conditions:

\[
f(S_0) = 0 \quad \text{and} \quad f'(S_0) = -\frac{2}{S_0}.
\]

In this case, the boundary conditions are chosen for our solution to match the level and slope of the static payoff function of the replication of geometric variance in the traded asset case, \( f(x) = -2\ln(x/S_0) \), at \( x = S_0 \).

Figure 3 shows the numerical solution to (29) for \( L=10 \), and Figure 4 shows how this solution differs from \( f(x) = -2\ln(x/S_0) \) for \( S_0 = 0.06 \).

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Figure 4, now at the end of the document, should be inserted here.

It is interesting to note that taking the limit \( L \to 0 \) in (28) and (29) leads to

\[
\frac{1}{2} f'' = 1,
\]

and

17
\[ \frac{1}{2} f'' = \frac{1}{x^2}, \]  

respectively, which are the ordinary differential equations that are solved by the quadratic and logarithmic functions that correspond to the traded asset cases of Section 3. This is a consequence of the fact that the gain accumulated at \( T_0 \) (24), for a traded swap of vanishing length, \( T_M - T_0 \), is linear in the increments in the underlying swap rate, leading to a replication problem mathematically equivalent to that in Section 3.

### 5.4 Implementation and valuation

In summary, the strategy that replicates arithmetic variance consists on holding a European contract with static payoff \( f_T \) that solves (30) and to execute, for the corresponding \( f' \), a dynamic trading strategy holding \( h_t = \frac{f'_S}{1-e^{-S_t(T_M-T_0)}} \) at-the-money swaps and reinvesting instantaneous dynamic gains via Libor deposits. The payoff function \( f \) and its derivative are computed only at the beginning of the variance swap and stored as a table for later use.

The identity (27) is a valid hedging strategy under Assumption 4.1. The execution of the strategy in reality involves investing in interest rate swaps and Libor deposits that are priced in the market without any approximation. Therefore we also test using

\[ \frac{f'_B(T_0)}{D_t} \]  

instead of \[ \frac{f'_S}{1-e^{-(T_M-T_0)S_t}} \] as the notional of the position in swaps. Notice that both quantities can be computed directly from the observable yield curve at \( t \).

The initial value of the fixed leg of the variance swap is equal to the sum of the initial values of European payoff and dynamic hedging rule. Because it only requires taking positions in at-the-money swaps that are worthless at inception, the dynamic rule has zero initial value. The initial value of the static payoff can be obtained by replication with call and put European options of the same maturity as in Carr and Madan (1998).

Finally, the replication and valuation of geometric variance follows the same steps described for arithmetic variance, but using the solution to (31) instead of (30).


6 Empirical testing

In this section we test the accuracy of the hedging strategy outlined in Section 5 using USD interest rate data. We are also testing, implicitly, the appropriateness of the diffusive dynamics adopted in (18) and the flat curve approximation (17) used in (24).

6.1 The data

We use daily interest rate data from December 1, 1997 to December 1, 2007, provided by Lehman Brothers. For each day in the sample, the set of rates is composed of: 3 month Libor, the first eight EuroDollar futures, and liquidly traded swap rates with tenors 2y, 3y, 5y, 10y, 15y, 20y and 30y. Rate levels are interpolated to construct a continuous instantaneous forward rate curve. Zero coupon bond prices, annuity factors, and forward swap rates are priced exactly from the continuous interest rate curve.

6.2 Testing strategy

We test the accuracy of the hedging strategy implied by (27) when $f$ is the solution of (30) or (31). We proceed in discrete time, using daily frequency for recording variance increments and rebalancing the dynamic portfolio. For convenience, we adopt a change of notation to measure time in days. In this case, for consecutive days $i = 0, ..., N$, (27) motivates a definition of dollar hedging error as the difference of hedge gains and realized variance

$$
\text{dollar error} \equiv f_N - f_0 - \sum_{i=0}^{N-1} \frac{f_i B_i(T_0)}{D_i} (V_{i+1}^i - V_i^i) - \sum_{i=0}^{N-1} (S_{i+1} - S_i)^2. \tag{34}
$$

The payoff function $f$ and its derivative are computed only once per variance swap and stored as a table. Then, having observed the full historical realization of forward curves for days $i = 0, ..., N$ we compute, for each day, the bond price $B_i(T_0)$, annuity factor $D_i$, the exact gain of the swap initiated the previous day, and the change in the swap rate over consecutive days. Then, the dollar hedging error in (34) can be computed exactly. Because market participants are used to quote prices in terms of implied volatilities, we take (34) as motivation, but actually compute and display errors in volatility terms.
We test the hedging of variance swaps of various lengths and underlying rates. The variance swaps we consider cover non-overlapping intervals for the recording of variance. Therefore, in 10 years of data we have 40 observations for variance swaps of length 3 months, and 10 observations for length 1 year.

6.3 Results

Figure 5 displays realized arithmetic variance against the gain delivered by the hedging strategy for a variance swap of length 3 months, defined on a forward swap rate of 5 years length. Each point in the plot is an observation, corresponding to a 3 month interval. Variances are multiplied by 4 to be annualized and by $10^4$ to be expressed in basis points. Moreover, we are hedging variance, but choose to display the square root of hedge gain against the square root of realized variance for the purposes of having volatility units that are closer to market quoting conventions. The square root of realized variance in Figure 5 ranges from 40 to 150 basis points per year, reflecting a wide dispersion in realized variance. The fact that all points are aligned near the line of slope 1 is evidence of the small hedging error relative to the stochasticity of realized arithmetic variance.

This size of the typical hedging error is reflected more precisely in Figure 6, showing the histogram of hedging errors (also in volatility units). The bid-ask spread of swaptions — the most liquid call and put options on swap rates — was between 1 and 5 basis points in volatility units in 2007, depending on the moneyness of the option, and higher in earlier years. Because the replication of a static European contract in terms of calls and puts as in Carr and Madan (1998) involves options of all strikes, and considering the relatively exotic nature of the variance swap, we can safely assume that 3 basis points in volatility is a conservatively small bid-ask spread for the price of a variance swap. The hedging errors in Figure 6 are much smaller than this, suggesting that the replication methodology is sufficiently accurate.
The importance of using a static payoff function that accounts for the additional variance related dynamic gains discussed in Section 5 is tested in Figure 7 where we repeat the discrete hedging test described above, but using \( f(x) = (x - S_0)^2 \) (both as static payoff and in determining \( f' \) that goes into the dynamic notional). Hedging errors in Figure 7 are one order of magnitude bigger than in Figure 6 and close to the natural bound suggested by the bid-ask spread of swaptions. We also test the dependence of the quality of the hedging methodology on the characteristics of the variance swap. Figure 8 shows hedging errors for a 1 year variance swap defined on a 10 year swap rate. A longer variance swap implies less non-overlapping periods, hence less observation points. Errors are comparable in magnitude to those in Figure 6.

We also test the replication of geometric variance for a 3 month variance swap on the 5 year swap rate. Figure 9 shows the square root of the gain of the hedging portfolio against the square root of realized geometric variance. We are normalizing variance in agreement with market conventions to obtain annualized Black volatilities. The wide dispersion of values for realized variance in Figure 9 suggests strong stochasticity for geometric realized variance.

A typical basis point volatility of 100 basis points per year, and interest rates at 5%, imply that the previously assumed bid-ask spread of 3 volatility basis points for the variance swap is equivalent to a 0.6 Black volatility bid-ask spread. Volatility hedging errors in Figure 10, for the 3m5y variance swap, and in Figure 11 for the 1y10y variance swap, are smaller than this, suggesting that the method is effective in hedging geometric variance. Tests for hedging geometric variance using the logarithmic function of the traded asset case from Section 3 lead to hedging errors (not shown) that are significantly worse than those that arise from using the solution of (31).
7 Conclusions

We have developed a replication and valuation methodology for arithmetic and geometric variance defined on a forward swap interest rate. The method uses a static payoff, characterized as the solution of an ordinary differential equation, and a dynamic rule implemented explicitly in terms of the most liquidly traded instruments according to current practice in the fixed income markets. Empirical results show that the hedging methodology is effective. The initial value of the fixed leg of the variance swap equals the initial value of the static contract, as the dynamic hedging rule is initially worthless.

References


Figure 1: Static payoff function for hedging arithmetic variance.
Figure 2: Ratio of payoff function for arithmetic variance and $f(x) = (x - S_0)^2$
Figure 3: Static payoff function for hedging geometric variance.
Figure 4: Ratio of payoff function for hedging geometric variance and $f(x) = -2\ln(x/S_0)$
Figure 5: Realized arithmetic variance vs. hedge gain, 3m5y
Figure 6: Histogram of hedging errors, arithmetic variance, 3m5y
Figure 7: Histogram of hedging errors, arithmetic variance with quadratic hedge, 3m5y
Figure 8: Histogram of hedging errors, arithmetic variance, 1y10y
Figure 9: Realized geometric variance vs. hedge gain, 3m5y
Figure 10: Histogram of hedging errors, geometric variance, 3m5y
Figure 11: Histogram of hedging errors, geometric variance, 1y10y