Variance swaps under no conditions

Conditional variance swaps are claims on realised variance that is accumulated when the underlying asset price stays within a certain range. Being highly sensitive to movements in both asset price and its variance, they require a very reliable model for pricing and risk-managing. Artur Sepp applies the Heston stochastic volatility model to derive closed-form solutions for pricing and risk-managing of such swaps requires intensive interpolation and extrapolation of implied volatility data in both strike and tenor dimensions, which is heavily bias-prone due to scarce and noisy market data. To get over these difficulties, we apply the Heston stochastic volatility model (1993) calibrated to be consistent with market prices of vanilla options for pricing and risk-managing of conditional variance swaps.

Notation
To characterise payouts of variance and conditional down-variance swaps, we introduce the following non-dimensional variables.

- Annualised one-period variance:

\[ V_n(t_{n-1},t_n) = A \left( \ln \frac{S(t_n)}{S(t_{n-1})} \right)^2 \]  (1)

where \( S(t) \) is the asset closing price observed at annualised times \( t_n \) (inception), \( \ldots, t_2 = T \) (maturity), \( \ln[S(t_n)/S(t_{n-1})] \) is a return realised over the time period \([t_{n-1}, t_n]\), and \( A \) is the annualisation factor (as a rule, \( A = 252 \) for the daily observation schedule).

- Annualised realised variance over time period \([t_0,T]\):

\[ I^V_n(t_0,T) = \frac{1}{N} \sum_{n=1}^{N} V_n(t_{n-1},t_n) \]  (2)

where \( N \) is a given number of observations (fixings) in the time period \([t_0,T]\).

- Annualised conditional down-variance:

\[ I^D_n(t_0,T) = \frac{1}{N} \sum_{n=1}^{N} \sum_{i=1}^{N} V_n(t_{i-1},t_i) 1_{[S(t_i) \leq L]} \]  (3)

where \( L \) is the down-barrier measured in the same currency unit as \( S(0) \).

- Occupation time below barrier \( L \):

\[ I^L_n(t_0,T) = \frac{1}{N} \sum_{n=1}^{N} 1_{[S(t_n) \leq L]} \]  (4)

Similarly to (3) and (4), we define conditional up-variance accrued when the spot is above the up-barrier \( U \) and occupational time above the barrier \( U \), respectively.

In continuous-time setting, assuming that the asset price, \( S(0) \), is driven by a diffusion process with instantaneous variance, \( \nu(0) \), whose dimension is \([\nu(0)] = \text{time}^{-1} \), these variables converge in probability to the following non-dimensional continuous-time variables:

Conditional variance swaps are recent financial innovations that enhance flexibility in volatility trading and risk management. They allow investors to take exposure to future market skew and convexity, and are attractive for investors with specific market scenarios because they are relatively inexpensive and flexible way to lock in funds. A recent article (Jung, 2006) indicates that there is growing interest in conditional and corridor variance swaps among hedge funds and proprietary desks.

In general, there are two main reasons for the popularity of conditional variance swaps: they allow an investor to take directional trades on both the future realised volatility and the level of the underlying asset and, at the same time, pay less than a plain variance swap; and they can reduce a seller’s sensitivity to large moves in spot price and volatility.

Due to significant exposure to market skew and convexity, conditional variance swaps require very careful pricing and risk-managing. Following the seminal work of Demeterfi et al (1999), who showed how to replicate a variance swap through a dynamic trading strategy involving a portfolio of call and put options with appropriately chosen weights, it is also possible to design a replication strategy for conditional variance swaps using a portfolio of European-style vanilla and digital options. An exposition of this topic is given by Youbi (2006).

However, as pointed out by Demeterfi et al, among others, constructing a replicating portfolio in practice is associated with certain limitations, including market liquidity and transaction costs. More importantly, for conditional swaps this replicating strategy...
where \( \hat{H} \) is the Heaviside step function, and \( \hat{I}(t,T) \) and \( \hat{H}(t,T) \) stand for annualised and de-annualised continuous-time quantities, respectively.

Discretisation bias introduced by switching to continuous-time quantities has been analysed in Sepp (2007), where Monte Carlo ties, respectively.

To generalise the above accumulative quantities, we introduce a new variable, denoted by \( I'(t,T) \), representing total accumulation over time period \([t,T]\) according to the contract function called the accumulator and denoted by \( g(S,V) \). The accumulator is assumed to be Markovian, that is, \( g(S,V) \) is a function of only the current values of \( S \) and \( V \). Then \( I'(t,T) \) at time \( t, t \in [t,T] \), can be represented as:

\[
I'(t,T) = \int_t^T g(S'(r),V'(r))dr = I'_0(t) + \int_t^T g(S'(r),V'(r))dr
\]

where \( I'_0(t) \) is a known quantity at time \( t \) and the last term represents future realisation. Here, \( I_0 \) and \( T \) are used as parameters and the current valuation time \( t \) is a variable.

We denote by \( R(t_0,t,T,S,V,I) \) (respectively \( P(t_0,t,T,S,V,I) \)) the undiscounted and de-annualised time-\( t \) value function of the floating (respectively fixed) leg of a swap on \( I' \), which is characterised according to its accumulator \( g(S,V) \). Under the risk-neutral valuation, the floating and fixed legs of the variance swap, denoted by \( R^u \) and \( P^u \), along with the corresponding legs of the down-variance swap, denoted by \( R^d \) and \( P^d \), are respectively given by:

\[
R^u(t_0,t,T,S,V,I) = \mathbb{E}^Q \left[ I'_T \mid \mathcal{F}(t) \right],
\]

\[
g^u(S,V) = V, \quad P^u(t_0,t,T,S,V,I) = 1,
\]

\[
R^d(t_0,t,T,S,V,I) = \mathbb{E}^Q \left[ I'^d \mid \mathcal{F}(t) \right],
\]

\[
g^d(S,V) = \hat{H}(L - S), \quad P^d(t_0,t,T,S,V,I) = \mathbb{E}^Q \left[ I'^d \mid \mathcal{F}(t) \right],
\]

where the expectation for time \( T \) is taken under the martingale (pricing) measure \( Q \) conditioned on the available information, \( \mathcal{F}(t) \), at time \( t \).

We denote the value function of a swap on \( I' \) by \( U(t_0,t,T,S,V,\theta_1,\theta_2,K) \), where \( K \) is the annualised delivery price per volatility point. Using (5), we represent value functions of the variance swap, \( U^u \), and down-variance swap, \( U^d \), respectively, as follows:

\[
U^u(t_0,t,T,S,V,I,K) = D(t,T) \left( \frac{1}{N} R^u(t_0,t,T,S,V,I) - K^2 \right)
\]

\[
U^d(t_0,t,T,S,V,I,K) = D(t,T) \left( \frac{1}{N} R^d(t_0,t,T,S,V,I) - K^2 \right)
\]

where \( D(S,T) \) is an appropriate discount factor, and for brevity we take the notional of a swap to be one currency unit.

To generalise notations, we denote the de-anualised and undiscounted value function of a fixed or floating leg of a swap on \( I' \) by \( U^u(t_0,t,T,S,V,\theta_1,\theta_2) \).

**Model**

We use the Heston (1993) stochastic volatility model and apply the augmentation procedure developed by Lipton (2001) to describe the joint evolution of \( S(t), V(t), I_0, I_0' \) under the pricing measure as:

\[
\frac{dS(t)}{S(t)} = (\mu - \delta(t))dt + \sqrt{V(t)}S(t)dW^S(t)
\]

\[
\frac{dV(t)}{V(t)} = \kappa(\theta - V(t))dt + \sqrt{V(t)}dW^V(t)
\]

\[
\frac{dI(t)}{I(t)} = g(S(t),V(t))dt, \quad I(t) = I_0, 0 \leq t \leq T
\]

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where \( r(t) \) and \( d(t) \) are the deterministic risk-free interest and dividend rates, respectively, \( \kappa \) is the mean-reverting rate, \( \theta \) is long-term variance, \( \varepsilon \) is the volatility of volatility, and \( W^r(t) \) and \( W^d(t) \) are correlated Wiener processes with constant correlation \( \rho \).

Applying the risk-neutral pricing and Feynman-Kac formula for the market model (7), we obtain the following pricing equation for \( Y(t_0, t, T, S, V, I) \):

\[
Y^S = \frac{1}{2} SV^2 \psi_S^{'^2} + (r(t) - d(t)) SY^S + \kappa (\theta - V) Y^S + \frac{1}{2} \varepsilon^2 V^2 \psi_{V}^{'}^2 + \rho \varepsilon V \psi_{V}^{'}^2 + g(S, V) Y^S = 0
\]

(8)

\[
Y^S(t_0,t,T,S,V,1) = I^S(t_0,T)
\]

**General solution**

To solve the pricing partial differential equation (8), we first introduce new variables \( S \rightarrow X = \ln \, S \), \( t \rightarrow T = t \), logarithmic barrier \( l = \ln \, L \), and new value function \( Y^0(t_0, t, T, S, V, I) \rightarrow Z^0(t_0, T, X, V, I) \), which solve the following problem:

\[
-Z^0_I + \frac{1}{2} VZ^0_{XX} + \left( r(T - t) - d(T - t) - \frac{1}{2} \varepsilon^2 V^2 \right) Z^0_X + \kappa (\theta - V) Z^0_X + \frac{1}{2} \varepsilon^2 V^2 Z^0_V + \rho \varepsilon V Z^0_V + g(X, V) Z^0_I = 0
\]

(9)

\[
Z^0(t_0,0,0,T,X,V) = I^S(t_0,0,T)
\]

The key observation here is that since the payout function is linear in \( I^S \), we can represent \( Z^0(t_0, t, X, V, I) \) as follows:

\[
Z^0(t_0,t,T,X,V,1) = I^S + \Xi^0(t_0,t,T,X,V)
\]

and obtain the following partial differential equation for \( \Xi^0(t_0,t,T,X,V) \):

\[
-\Xi^0_I + \frac{1}{2} V\Xi^0_{XX} + \left( r(T - t) - d(T - t) - \frac{1}{2} \varepsilon^2 V^2 \right) \Xi^0_X + \kappa (\theta - V) \Xi^0_X + \frac{1}{2} \varepsilon^2 V^2 \Xi^0_V + \rho \varepsilon V \Xi^0_V + g(X, V) = 0
\]

(9)

\[
\Xi^0(0,0,T,X,V) = 0
\]

This equation can be solved by applying the celebrated Duhamel’s principle, which states that the solution of an inhomogeneous parabolic problem with a zero initial condition can be represented as the convolution of the source term with the Green’s function, \( \hat{G}(t, T, X, V, \Theta) \), solving the corresponding homogeneous partial differential equation (9) with initial condition \( \hat{G}(0, T, X, V, \Theta) = \delta(X - X^0) \delta(V) \).

\[
\Xi^0(t_0,t,T,X,V) = \int_{t_0}^{t} \Sigma(t',T,X,V) dt'
\]

(10)

where:

\[
\Sigma(t',T,X,V) = \int_{t_0}^{t} \int_{t_0}^{t} \hat{g}(t',T,X,V,X',V') G(t',T,X,X',V,V') \, dx \, dv
\]

(11)

and we assume that the source function \( \hat{g}(x, V) \) vanishes for \( V < 0 \).

Now, \( \Sigma(y,t,T,X,V) \) can be calculated by applying the generalised Fourier transform in \( X \) and \( V \), with respective transform variables \( \Phi = \Phi_X + i \Phi_V \) and \( \Theta = \Theta_X + i \Theta_V \), where \( i = \sqrt{-1} \) and \( \Phi_X, \Phi_V, \Theta_X, \Theta_V \in \mathbb{R} \), and employing the convolution formula for the Fourier transform:

\[
\hat{g}(t',T,X,V) = \frac{1}{4 \pi} \int \mathcal{R} \left[ \hat{g}(\Phi, \Theta) \hat{G}(t',T,X,V,\Theta) \, d\Phi \, d\Theta \right]
\]

(12)

where \( \hat{g}(\Phi, \Theta) \) is the generalised Fourier transform of the source term:

\[
\hat{g}(\Phi, \Theta) = \int e^{i \Phi X + i \Theta V} \hat{g}(X, V) dX dV
\]

(13)

and \( \hat{G}(t, T, X, V, \Theta) \) is the kernel of the generalised Fourier transform of the Green’s function for joint evolution of \( X \) and \( V \) in the Heston model given by:

\[
\hat{G}(t, T, X, V, \Theta) = e^{-\theta X + \frac{\theta^2}{2} t} \left( e^{i \theta V} + C \right)
\]

(14)

An outline of the derivation of formula (14) is given in the appendix.

Examining formula (14), we note that the real parts of the expressions inside the square root and the logarithm are positive if:

\[
-\frac{1}{1 - \rho^2} < \Phi_X < \frac{1}{1 - \rho^2}; \quad \Theta_V > -\frac{2 \kappa}{\epsilon^2}
\]

so that \( A(t, \Phi, \Theta) \) and \( B(t, \Phi, \Theta) \) are continuous functions in the above range.

We also introduce the following function:

\[
\hat{G}_\Theta(t, T, X, V, \Theta) = \frac{-i}{\Theta} \hat{G}(t, T, X, V, \Theta)|_{\Theta = 0}
\]

(15)

\[
= (A_0(t, \Phi) + B_0(t, \Phi)V) \hat{G}(t, T, X, V, 0)
\]

where:

\[
A_0(t, \Phi) = 2 \Phi_X \left[ 1 - e^{-\frac{\theta X}{\epsilon^2}} - \frac{\theta X}{\epsilon^2} - \psi_+ \right]
\]

\[
B_0(t, \Phi) = \psi_+ \left[ e^{-\frac{\theta X}{\epsilon^2}} + \psi_- \right]
\]

As a result of our development, we calculate the de-annualised and undiscounted value of a floating or fixed leg of a swap on \( I' \) in the following way:

\[
Y^S(t_0,t,T,S,V,1) = I^S_0 + \Xi^0(T - t, T, \ln S, V)
\]

(16)
Implementation and results  
We note that the pricing formula (16) is a generic result and it can be applied for swaps with general Markovian accumulators (S, V). To implement this formula, we proceed as follows:  
- For a specified accumulator function g(e^x, V), we find its Fourier transform (13).  
- For a given value of τ, we evaluate the value function (12) by doing a two-dimensional (in some cases, one-dimensional) numerical inversion of the Fourier transform.  
- We calculate the time-integral in (10) using an appropriate numerical integration scheme.  

We note that, although feasible, the two-dimensional inversion of the Fourier transform is somewhat cumbersome to deal with, so we reduce the dimension of the problem at hand as much as possible. Next, we consider the last two steps in more detail, and then develop some analytical reductions for variance and conditional variance swaps.  

Numerical inversion. First, we need to ensure that the kernel of the Green’s function (14) is a decaying function of Φ and Θ, and the exponent has a linear decay in Φ, and only logarithmic decay in Θ. Accordingly, the integral in formula (12) is a well-defined function of Φ and Θ.  

Limits (17) can serve as a means to establish fixed integration bounds corresponding to a desired level of accuracy. Alternatively, for numerical integration we can use floating bounds with an appropriate locally stopping rule. We suggest the Simpson’s rule for numerical integration we can use floating bounds with an appropriate representation of its value function. Without introducing technicalities, we assume that the operations of integration and differentiation are interchangeable, and we will derive its detailed expression using (27) along formula (28) for complex-valued exponentials.

Numerical inversion. First, we calculate the transform (13) of the accumulator function of the floating leg as follows:

\[ \hat{g}^{\text{cov}}(\Phi, \Theta) = \int \int e^{i \Phi \tau + i \Theta V} dX'dV'd\tau' = 2\pi \delta_0(\Phi) [ \int e^{i \Theta V} dV'] = 4\pi^2 \delta_0(\Phi) \delta_0(\Theta) \]

Applying the above result to formula (12), we obtain:

\[ \Sigma^{\text{cov}}(\tau, T, X, V) = \mathbb{R} \left[ \hat{G}_0(\tau, T, X, V) \right]_{\Phi=0} = V e^{\tau x} + \left( \frac{1}{k} - e^{-\tau x} \right) \theta \]

and by calculating (10) we derive the value function of the floating leg:

\[ \Xi^{\text{cov}}(\tau, T, X, V) = \int_0^T \Sigma^{\text{cov}}(\tau', T, X, V) d\tau' = \frac{1}{k} \left( 1 - e^{-\tau x} \right) V + \frac{\theta}{k} (\theta x + e^{\tau x} - 1) \]

As a result, the value function of the variance swap is given by:

\[ U^{\text{cov}}(t_0, t, T, S, V, I^8, K) = D(t, T) \left( \frac{AF}{N} \left( I_0^8 + \Xi^{\text{cov}}(T - t, T, \ln S, V) \right) - K^2 \right) \]

which is in agreement with the well-known formula for the value of variance swap under the Heston model obtained, for example, by Lipton (2001) using a different approach.

Conditional down-variance swap. First, we consider the floating leg of the down-variance swap and calculate its transformed accumulator as follows:

\[ \hat{g}^{\text{cov}}(\Phi, \Theta) = \int \int e^{i \Phi \tau + i \Theta V} \mathcal{H}(l - X') V dX'dV' = 2\pi \frac{e^{\Phi \theta}}{\Phi} \delta_0(\Theta) \]

provided \( \Phi > 0 \).  

Thus formula (12) becomes:

\[ \Sigma^{\text{cov}}(\tau, T, X, V) = \frac{1}{\pi} \int_0^\infty \mathbb{R} \left[ \frac{1}{\Phi} e^{i \theta} \hat{G}_0(\tau, T, X, V, \Phi) \right] d\Phi \]

and the value function of the floating leg is given by:

\[ R^{\text{cov}}(t_0, t, T, S, V, I^8) = I_0^8 + \int_0^T \Sigma^{\text{cov}}(T - t', T, X, V) d\tau' \]

Similarly, for transform (13) of the accumulator of the fixed leg we obtain:

\[ \hat{g}^{\text{cov}}(\Phi, \Theta) = \int \int e^{i \Phi \tau + i \Theta V} \mathcal{H}(l - X') V dX'dV' = 2\pi \frac{e^{\Phi \theta}}{\Phi} \delta_0(\Theta) \]

provided \( \Phi > 0 \).  

Then, applying formula (12), we obtain:

\[ \Sigma^{\text{cov}}(t, T, X, V) = \frac{1}{\pi} \int_0^\infty \mathbb{R} \left[ \frac{1}{\Phi} e^{i \theta} \hat{G}_0(\tau, T, X, V, 0) \right] d\Phi \]

and, as a result, for the fixed leg we derive the following formula:

\[ P^{\text{cov}}(t_0, t, T, S, V, I^8) = I_0^8 + \int_0^T \Sigma^{\text{cov}}(T - t', T, X, V) d\tau' \]

The value function of the conditional down-variance swap is then calculated using the formula given in (6). Finally, we note that the symmetry of the floating and fixed legs of the up-variance swap are calculated using formulas (20) and (21) respectively with the following substitutions: \( \Phi \rightarrow -\Phi \) and \( l \rightarrow u \), where \( u = \ln U \) is the logarithmic up-barrier.  

Numerical evaluation of formulas (20) and (21) is no more complicated than calculating a European-style option price under the Heston model, and the time-integral is easy to handle.

Results and discussions. Variance swaps are typically charaterised by their fair strike. The fair strike is chosen in such a way that the total value of a swap at inception (\( t = t_0 = 0 \)) is zero. Using our results, we find that the fair strikes of the variance swap, \( K^\alpha \), and down-variance swap, \( K^\alpha \), with maturity \( T \) are respectively given by:

\[ K^\alpha = V e^{\tau x} + \left( \frac{1}{k} - e^{-\tau x} \right) \theta \]

\[ K^\alpha = \frac{1}{k} \left( 1 - e^{-\tau x} \right) V + \frac{\theta}{k} (\theta x + e^{\tau x} - 1) \]
Cutting Edge. Variance Swaps

1 Term structure of fair strikes

Fair strikes of variance swap (Strike RV), down-variance swap with barriers $L = 0.95$ and $L = 0.9$ (Strike DV1 and Strike DV2), and up-variance swap with barriers $U = 1.05$ and $U = 1.1$ (Strike UV1 and Strike UV2) as functions of maturity. Relevant contract and model parameters are: $S(0) = 1.0, A/N = 1/7, r(t) = 0, c = 0, V(0) = 0 = 0.3, k = 4.0, \varepsilon = 0.8, \rho = -0.8$.

$$K^{DV} = \left(\frac{A}{N}\left(1 - e^{-\varepsilon t}\right) \frac{1}{V} + \frac{\theta}{N}\left(T_1 + e^{-T_1 -1}\right)\right)^{1/2}$$

$$K^{UV} = \left(\frac{A}{N}\left(T - e^{-\varepsilon t}\right) \frac{1}{V} + \frac{\theta}{N}\left(T_1 + e^{-T_1 -1}\right)\right)^{1/2}$$

The fair strike of the up-variance swap is calculated in a similar fashion.

In figure 1, we show the term structure of the fair strikes of variance swaps and conditional down- and up-variance swaps. To exclude the mean-reversion effect, we use $V(0) = 0$, so that the term structure of the variance swap is flat.

In general, in the presence of negative, $\rho < 0$ (positive, $\rho > 0$), skew, the following conclusions follow: the fair strike of down-variance is above (below) the fair strike of the variance swap and it is increasing (decreasing) with a decreasing (increasing) barrier level; and the fair strike of up-variance is below (above) the fair strike of the variance swap and it is decreasing (increasing) with an increasing (decreasing) barrier level.

In figure 2, we show the fair strike of a down-variance swap as a function of $\rho (\text{Rho})$ and $\varepsilon (\text{Volvol})$. The opposite is that a plain variance swap yields exposure only to the term structure of realised variance while a conditional variance swap gives exposure to both skew and convexity. Making $\rho$ and $\varepsilon$ time-dependent allows us to introduce more flexibility in modelling forward volatility surfaces and, as a result, in modelling conditional variance swaps. A detailed account of such a model is given in Sepp (2006).

Finally, we notice that the accumulator of a conditional variance swap is somewhat linear in the daily realised variance, and further optionality can be introduced by using local cap and floor levels on the daily realised variance.

Extensions

We briefly discuss some extensions of the framework we laid out.

Some contracts on conditional variance specify that the barrier is set on the previous fixing price, that is, that the down-variance, $I_{DN}^{dv}(t_0, T)$, is accrued according to:

$$I_{DN}^{dv}(t_0, T) = \sum_{n=1}^{N} \left[ \ln \left( \frac{S(t_n)}{S(t_{n-1})} \right) \right] ^{2} \left[ \delta(t_n) \right]$$

To apply our results to the floating leg of this swap, we use the following approximation:

$$I_{DN}^{dv}(t_0, T) = \frac{1}{N} \sum_{n=1}^{N} V_n(t_n, t_{n+1}) [S(t_n) \delta(t_n)]$$

When variance dynamics do not depend on the spot price, which is a typical assumption, the key difference between a conditional variance swap and an option on a variance swap is that the latter produces only exposure to convexity, measured by $\varepsilon$, and no exposure to skew, measured by $\rho$. The opposite is that a conditional variance swap gives exposure to both skew and convexity. In addition, we note that a plain variance swap yields exposure only to the term structure of realised variance while both options on variance swaps and conditional variance swaps are also exposed to the term structure of realised variance.

Furthermore, the time-integral in formulas (20) and (21) indicates clearly that conditional variance swaps are exposed to future levels of market skew and convexity. Making $\rho$ and $\varepsilon$ time-dependent allows us to introduce more flexibility in modelling forward volatility surfaces and, as a result, in modelling conditional variance swaps. A detailed account of such a model is given in Sepp (2006).

We note that in this case the accumulator, $g(t, V)$, is no longer Markovian with dependence on the last closing price.
Appendix: derivation of formula (14)

We solve the Green’s function associated with the homogeneous partial differential equation (9) by assuming that it is given by an affine form:

\[
G(\tau, T, X, \Phi, V^r, V^i) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{\Phi X + \frac{\Phi^2}{2}} \int_{-\infty}^{\infty} e^{-\Phi V^r + \frac{\Phi^2}{2}} \mathcal{G}(\tau, T, X, \Phi, V^r, V^i) d\Phi d\Theta_t
\]

where:

\[
\mathcal{G}(\tau, T, X, \Phi, V^r, V^i) = e^{-\Phi X + \frac{\Phi^2}{2}} \left( e^{-\Phi V^r + \frac{\Phi^2}{2}} + e^{-\Phi V^i + \frac{\Phi^2}{2}} \right)
\]

is the kernel of the Fourier transform with \(A(0, \Phi, \Theta) = 0\) and \(B(0, \Phi, \Theta) = -\Theta\).

Plugging (24) into partial differential equation (9), we obtain the following system of ordinary differential equations:

\[
\begin{align*}
-\frac{1}{2} \frac{\partial^2 G}{\partial X^2} - (\kappa + \epsilon \Phi) B + \frac{\partial}{\partial \Theta} \left( \frac{\partial G}{\partial \Theta} \right) &= 0, \\
B(0, \Phi, \Theta) &= -\Theta \\
A(0, \Phi, \Theta) &= 0
\end{align*}
\]

and

\[
-\frac{1}{2} \frac{\partial^2 G}{\partial \Theta^2} + \frac{\partial}{\partial \Theta} \left( \frac{\partial G}{\partial X} \right) &= 0
\]

The equation for \(A\) is solved by introducing function \(C(\tau, \Phi, \Theta)\) such that \(C(0, \Phi, \Theta) = -e^{\mathcal{C}BC/2}\) with initial conditions \(C(0, \Phi, 0) = 1\) and \(C(0, \Phi, \Theta) = -e^{\mathcal{C}BC/2}\) with initial conditions \(C(0, \Phi, 0) = 1\) and \(C(0, \Phi, \Theta) = -e^{\mathcal{C}BC/2}\).

\[
A\left(\tau, T, \Phi, \Theta\right) = -e^{\mathcal{C}BC/2} \left( e^{-\epsilon \Phi V^r + \frac{\Phi^2}{2}} + e^{-\epsilon \Phi V^i + \frac{\Phi^2}{2}} \right)
\]

where \(\mathcal{C} = 1/4\) and we used formula (18) for the expected variance at time \(T\) given \(\mathcal{F}(t)\). For the fixed leg of this contract, we use \(I^\mathcal{C}(t_0, T) = I^\mathcal{C}(t_0, T - \delta\tau)\).

For pricing variance swaps conditioned on the spot price, staying above the down-barrier \(L\) or below the up-barrier \(U\) or within the corridor \(L < S < U\), we construct replicating portfolios that consist of variance swaps, conditional down-variance and up-variance swaps using our derived formulas.

Our results can be extended in a straightforward way to the Heston model with volatility jumps, as studied in detail by Sepp (2007). In general, the key is to have a closed-form expression for the generalised Fourier transform of the Green’s function associated with a given stochastic volatility model.

Time-dependent barrier levels \(L(t)\) and \(U(t)\) can be handled by introducing appropriate variables. For example, in the case of a down-barrier specified by the exponential form \(L(t) = L_0 e^{\alpha t}\), we use the new variable \(X = \ln(S/L_0 - \alpha t)\) to reduce the problem to one with a flat barrier.

Given the affine structure of our solution, calculating important risk parameters, including deltas and gammas with respect to the spot and variance, can be done by differentiating the Green’s function and inverting the corresponding transforms numerically. In this respect, having a closed-form solution for values of a conditional variance swap is a great advantage over the Monte Carlo method.

Conclusions

We have considered the pricing problem of conditional variance swaps, which are an innovation in the business of volatility trading. We applied the Heston stochastic volatility model for describing the joint evolution of the asset price and its variance, and derived closed-form solutions for the pricing problem of conditional variance swaps. One of our key results, pricing formula (16), can easily be extended to value swaps with general Markovian accumulators. The obvious advantage of having closed-form pricing formulas is the ability to calculate risk parameters in a fast and reliable way.

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