Smile dynamics III

In two articles published in 2004 and 2005 in Risk, Lorenzo Bergomi assessed the structural limitations of existing models for equity derivatives and introduced a new model based on the direct modelling of the joint dynamics of the spot and the implied variance swap volatilities. Here he presents new work on an extension of this model, which, while remaining Markovian, provides control on the smile of forward variances and can be calibrated to Vix futures and options.

To remedy some of the limitations of popular models used for equity derivatives highlighted in previous work (Bergomi, 2004), we proposed a stochastic volatility model based on a specification of the joint dynamics of the underlying spot and its implied forward variance swap (VS) variances (Bergomi, 2005). The aim of this model was to afford better control of:

- the term structure of the volatility of volatility;
- the forward skew; and
- the correlation between spot and VS volatilities

for the pricing of options such as reverse cliquets, accumulators and options on realised variance. In practice, as there were no traded instruments to hedge the volatility-of-volatility risk, the level of the corresponding parameter in our model was set to a conservative value and we concluded our article with the following statement: “It is the hope of the author that the liquidity of options on volatility and variance increases so that we will soon be able to trade the smile of volatility of volatility!”

The author’s hopefulness was fulfilled, as these past few years have witnessed the growth of the market in options on realised variance and options on realised variance. In this article, we propose a new version of our model that addresses the issue of the smile of volatility of volatility, which manifested, for example, in the smiles of options on Vix futures.

We first present our model in its two forms, continuous and discrete, and motivate the use of either one. We then illustrate its application to the market of Vix futures and options, as well as options on realised variance, using the continuous version. We next examine some properties of the vanilla smile that this model produces and summarise our work in the concluding section.

Dynamics for forward variances

In previous work (Bergomi, 2005), from which we borrow both vocabulary and notation, we proposed two versions of a model aimed at modelling either (a) a set of discrete forward variances or (b) the full variance curve. The use of a discrete structure was then motivated by the need to separately control the short forward skew and the spot-volatility correlation. Our starting point was the following lognormal dynamics:

\[ \xi_T^f = \xi_0^f \exp \left( \omega \sum_{n=1}^{\infty} w_n e^{-k(T-t)} X_n \right) - \frac{\omega^2}{2} \sum_{n=1}^{\infty} w_n^2 e^{-2(k+\kappa)(T-t)} E[X_n X_m] \]  

for either:

- continuous forward variances \( \xi^f_T \); the forward instantaneous variance for date \( T \) observed at \( t \); or
- discrete forward variances \( V_{T_n, T_{n+1}} \); we first set up a tenor structure of dates \( T_n \), \( V_{T_n, T_{n+1}} \) is then the forward variance for the time interval \( [T_n, T_{n+1}] \), observed at \( T_n \).

The initial values of forward variances, \( \xi^f_0 \), are inputs of the model, just like yield curves in interest rate models. The \( X_n \) are correlated Ornstein-Uhlenbeck processes:

\[ dX_n = -(k_1 + k_2) X_n dt + dW^n_\tau \]

where \( k_1 > k_2 \), \( X_0 = Y_0 = 0 \), \( W^n_\tau \) and \( W^\tau \) are correlated with each other as well as with \( W \), the Brownian motion driving the spot process, which will be specified in due course. The correlations are defined as \( \langle dW^n dW^\tau \rangle = \rho_{n\tau} dt \), \( \langle dW dW^\tau \rangle = \rho_{n\tau} dt \).
\[ \rho_{st} dt \] with the following parameterisation:
\[ \rho_{st} = \rho \rho_{sx} + \chi \sqrt{1 - \rho^2} \sqrt{1 - \rho_{sx}^2} \] (2)\n\[
X_t \text{ and } Y_t \text{ and their increments are easily simulated using their integral representation:}
\[ X_t = \int_0^t e^{-k(T-s)} dW_s^X, \quad Y_t = \int_0^t e^{-k(T-s)} dW_s^Y. \] (3)
Choosing to model discrete or continuous forward variances is a matter of practical convenience, motivated by the nature of the financial observables one needs to model.\(^2\) While Vix futures naturally call for a discrete framework, the continuous form is more suited to options involving implied or realised variances of arbitrary maturities. In what follows we develop both types of model, starting with the continuous one.

**A Markovian model for continuous forward variances.** We take as our starting point equation (1) – with two factors – but relax lognormality while keeping the model Markovian. Let us define \( x^T_t \) as:
\[ x^T_t = \alpha_k \left[ (1-\theta) e^{-k(T-t)} X_t + \theta e^{-k(T-t)} Y_t \right] \] (4)
where:
\[ \alpha_k = 1/ \sqrt{(1-\theta)^2 + \theta^2 + 2\rho(1-\theta)} \]
\( x^T_t \) is driftless by construction. Let us now introduce a function \( f^T(x, t) \) that maps \( x^T_t \) on to the forward variance \( \xi^T_t \):
\[ \xi^T_t = \xi^T_T f^T(x^T_t, t) \] (5)
The condition that \( \xi^T_t \) be driftless translates into the following condition for the mapping function \( f^T(x, t) \):
\[ \frac{d f^T}{dt} + \frac{\sigma^2(t)}{2} \frac{\partial^2 f^T}{\partial x^2} = 0 \] (6)
where \( \sigma(t) \) is given by:
\[ \sigma^2(t) = \alpha_k^2 \left[ (1-\theta)^2 e^{-2kT} + \theta^2 e^{-2kT} + 2\rho(1-\theta) e^{-k(t+T)} \right] \] (7)
The normalisation factor \( \alpha_k \) has been introduced so that \( \sigma(0) = 1 \). A solution to equation (6) needs to be properly rescaled so that \( f^T(0, 0) = 1 \).
We also require that \( f^T(x, t) \) be monotonic in \( x \). The lognormal case (equation (2) in Bergomi, 2005) corresponds to the following particular solution of equation (6):
\[ f^T(x, t) = e^{-\frac{\sigma^2(t)}{2} x^2} \Theta(x) \] (8)

Mapping a Gaussian process on to a financial instrument is a popular technique in fixed income. In ‘Markov-functional models’ (Kennedy, Hunt & Pelsser, 2000), \( f \) maps a Gaussian process on to a Libor or swap rate setting at \( t = T \). \( f^T(x, t = T) \) is chosen so that the market caplet or swaption smile is calibrated. Different Libor rates of the same yield curve can be driven either by different processes or by the same process, although this may generate peculiar behaviour.\(^3\)

Using the ansatz \( S_t = f(W_t, t) \), where \( W_t \) is a Brownian motion, for equity underlyings is not as popular. Indeed, in the equity context one requires that the model match market smiles for several maturities, for the same underlying. This cannot be achieved by writing \( S_t = f(W_t, t) \) as the solution of equation (6) is fully determined by its terminal condition at \( t = T \), set by the market. Smiles for maturities shorter than \( T \) generated by equation (6) will generally not agree with market smiles. For examples of Markov-functional models in the equity context, see Carr & Madan (1999).

Markov-functional models are local volatility models whose local volatility function is set by the mapping function \( f \). In the case where \( x \) is a Brownian motion, it is given by:
\[ \sigma(t, S) = \frac{\partial f}{\partial x \mid_{x = f^{-1}(S, t)}} \]
They are not suitable for pricing options with high sensitivity to forward volatility or volatility of volatility, but are an economical solution to the issue of single-maturity smile modelling. This is the case for Vix options, whose maturities are also the expiry dates of the underlying futures.

The case for local volatility can in fact be argued much more convincingly for volatility than for equities themselves: from 1920 to 2000, the Dow Jones index rose by a factor of around 100, while volatility’s order of magnitude has changed little in the past two centuries (the average volatility of the Dow Jones index from 1920 to 2000 was 16%, while the volatility of an index built using stocks trading on the Paris bourse from 1801 to 1900 was around 14%).

Once \( f^T(x, t = T) \) has been chosen, solving equation (6) generates \( f \) for times \( t < T \). This has to be done once for each \( T \). One is then able, given the values of \( X_t \) and \( Y_t \) at time \( t \), to generate the full variance curve \( \xi^T_t \).

However, having to solve equation (6) for many values of \( T \) is impractical on one hand, and on the other hand, traded instruments provide information on discrete forward variances:
\[ \nu^T_{t+1} = \frac{1}{T_{t+1} - T_t} \int_{T_t}^{T_{t+1}} f^T_{t+1}(x, t) \, dt \]
rather than instantaneous ones; Vix futures, for example, are related to one-month forward variances. We now present two solutions to this issue that can be used practically.

**A single \( f^T \) per time interval.** Define a tenor structure \( T_j \) for example given by the maturities of Vix futures, and assume that all functions \( f_j \) are equal to \( f^T \) for \( T \in [T_j, T_{j+1}] \):
\[ f^T(x, t) = f^T(x, t) \]
Since \( f^T(x, t) \) is defined for \( t < T \), only, we then need to specify how \( \xi^T_t \) evolves to \( \xi^T_{t+1} \). We can simply assume that \( \xi^T_t \) stays frozen: \( \xi^T_{t+1} = \xi^T_t \). We then need to generate one mapping function \( f^T(x, t) \) per interval \([T_j, T_{j+1}]\).

**An analytical ansatz for \( f^T \).** Exponentials are eigenfunctions of the heat equation (6) and generate the lognormal solution (8). The familiar representation of space-time harmonic functions suggests that we use as an ansatz for \( f^T \) a linear combination of exponentials:
\[ f^T(x, t) = \int_0^\infty d\mu(\alpha) e^{-\frac{\sigma^2(t)}{2} \alpha^2} \Theta(x) \] (9)
where \( d\mu(\alpha) \) is a measure such that \( \int_0^\infty d\mu(\alpha) = 1 \). We expect that mixing several exponentials with positive weights will generate a positively sloping volatility, which is what Vix smiles exhibit. Let us use a combination of two exponentials and write:
\[ f^T(x, t) = e^{-\alpha x^2} \]
^1 In fixed income, one similarly chooses to model either continuous forward rates or discrete Libor rates.
^2 For example, it was shown (Kennedy, Hunt & Pelsser, 2000) that in a Markov functional model forward rates cannot be lognormal. This, however, does not mean that a Markov functional model is incapable of generating flat smiles for Libor rates. It actually will, but while Libor smiles are flat, the associated local volatilities are not flat, except for the particular Libor rate whose numerator is chosen as how numerator.
^3 Calculated using monthly data provided in Gallio & Henss (2007)
The spot process in the discrete version.

\[
 f^T(x,t) = \left(1 - \gamma_T\right) e^{\omega_T x} e^{-\frac{\omega_T x^2}{2}} + \gamma_T e^{\omega_T x} e^{-\frac{\omega_T x^2}{2}}
\]

where \( h(t,T) = f_t, \sigma(t) dt \) and \( \sigma(t) \) is defined in equation (7). \( \gamma_T, \omega_T \) are functions of \( T \). \( \omega_T \) is a scale factor for the volatility of \( \xi_T^t \) let us write it as:

\[
 \omega_T = 2 \nu \frac{\xi_T^t}{1 - \gamma_T + \gamma_T B_T}
\]

where we have introduced \( V \) and \( \xi_T^t \) controls the general level of volatility of volatility in the model while \( \xi_T^t \) is an adjustment factor, controlling the volatility of \( \xi_T^t \). Using these new variables, the dynamics of \( \xi_T^t \) at time \( t = 0 \) reads:

\[
d\xi_T^t = 2V \xi_T^t q_t^x dt
\]

For \( T \to 0 \), \( \xi_T^t \) has an instantaneous volatility equal to \( 2V \xi_T^t \). Ideally, we would like the \( \xi_T^t \) to be close to one: \( V \) is then the volatility of a very short VS volatility.

For the sake of calibrating \( \gamma_T, \omega_T, \xi_T^t \) to Vix futures and Vix options expiring at time \( T \), we will take them to be piece-wise constant over the interval \([T_i, T_{i+1}]\), equal to \( \gamma_T, \omega_T, \xi_T^t \).

### The spot process in the continuous version.

The dynamics for \( S \) is given by:

\[
dS = (r - q)Sdt + \sqrt{V_T} \sqrt{S} \sqrt{f^T(x,t)} dW_t
\]

Note that correlations \( \rho_{S,v} \) and \( \rho_{S,W} \) of \( W \) with, respectively, \( W^v \) and \( W^w \) will determine both the correlation between the spot and VS volatilities as well as the skew of the vanilla smile generated by the model.

Calibration and pricing results obtained using the analytical ansatz for \( f^T \) in the continuous version of our model will be discussed below. Prior to this, let us present the discrete form of our model.

### A Markovian model for discrete forward variances.

If we are only interested in modelling the joint dynamics of the S&P 500 and Vix futures, rather than the full dynamics of the S&P 500 variance curve, it is natural to define a tenor structure of equally spaced dates \( T_i \) corresponding to the monthly expiries of Vix futures and to directly model discrete forward variances \( V_T^{T,T_{i+1}} \).

Mirroring the dynamics for instantaneous variances, let us define processes \( x_i \) and write:

\[
 V_T^{T,T_{i+1}} = \frac{V_0^{T,T_{i+1}}}{V_0^{T,T_i}} f^T(x_i,t)
\]

\[
 x_i = \alpha \left(1 - \theta \right) e^{-h(T,T_i)} X_t + \theta e^{-h(T,T_i)} Y_t
\]

\[
 V_0^{T,T_i} \text{ are inputs given by the VS market. The mapping functions } f^T \text{ satisfy the following equation:}
\]

\[
 \frac{df^T}{dt} + \frac{\sigma^2(T,T_i)}{2} \frac{df^T}{dx^2} = 0
\]

where \( \sigma(T) \) is defined in (7).

One of the benefits of using the discrete framework is that calibration to the smiles of Vix options is exact. To generate the terminal values of the \( f(x,t \to T_i) \) so that market prices of Vix futures and options are matched, use the following simple procedure based on Kennedy, Hunt & Pelsser (2000).

1. Compute the level of forward variance \( V_0^{T,T_{i+1}} \) from the Vix future and options of maturity \( T_i \). The corresponding expression is given in equation (13) further below.
2. From the Vix smile for maturity \( T_i \), one easily generates the price of a digital option of maturity \( T_i \) on Vix future \( F_i \) that pays one if \( F_i \) is below a barrier \( L \). This is also the price of the same digital on \( V_{T_i}^{T,T_{i+1}} \), but with barrier \( L = F_i \). The price of this digital in our model is \( \mathcal{N}(x'(L)) \), where \( x'(L) \) is defined by:

\[
 L = V_0^{T,T_{i+1}} f^T \left(x'(L), T_i \right)
\]

and where \( \mathcal{N} \) is the cumulative density for \( x' \), a centred Gaussian random variable with variance \( (1 - \theta)^2 E[X_{T_i}^2] + \theta^2 E[\gamma_T^2] + 2 \theta (1 - \theta) E[X_{T_i} \gamma_T] \). Equating the model price to the market price determines \( x'(L) \). We now have one value for \( f^T \) at point \( x = x'(L) \):

\[
 f^T \left(x'(L), T_i \right) = \frac{L}{V_0^{T,T_{i+1}}}
\]

Doing this for sufficiently many values of \( L \) completely specifies the function \( f(x, t \to T_i) \). Equation (12) generates \( f^T \) for \( t < T_i \).

### The spot process in the discrete version.

Let us start with the following dynamics for \( S \) on the interval \([T_i, T_{i+1}]\):

\[
dS = (r - q)Sdt + \sqrt{V_T} \sqrt{S} dW_t
\]

This dynamics for \( S \) is lognormal over \([T_i, T_{i+1}]\). Just like in the continuous version of the model, the level of both forward skew and vanilla skew as well as the correlation between \( S \) and VS volatilities will be controlled by \( \rho_{S,v} \) and \( \rho_{S,W} \) the correlations of \( W \) with \( W^v \) and \( W^w \).

The discrete framework, however, brings us the additional capability of being able to control the short forward skew independently. We introduce, as in Bergomi (2005), a ‘local volatility’ function:

\[
 \sigma_T \left( \frac{S}{S_T}, V_T^{T,T_{i+1}} \right)
\]

and write the dynamics of \( S \) over \([T_i, T_{i+1}]\) as:

\[
dS = (r - q)Sdt + \sigma_T \left( \frac{S}{S_T}, V_T^{T,T_{i+1}} \right) \sqrt{S} dW_t
\]

where \( \sigma_T \) is defined so that:

- the VS volatility for maturity \( T_{i+1} \) observed at \( T_i \);}

\[
 \sqrt{V_T^{T,T_{i+1}}} \sqrt{V_T^{T,T_{i+1}}}
\]

is matched; and

- the desired level of forward skew and its dependence on the level of volatility are obtained. For example, it is easy to generate a short forward skew whose level is independent of or proportional to the level of short forward at-the-money volatility. Any type of dependence is easily accommodated. For details on the procedure for determining \( \sigma_T \), see Bergomi (2005).

We now have a model that calibrates exactly to the Vix market and affords full control of the short forward skew.

To calibrate the vanilla smile of the S&P 500 in addition to the Vix market, choose the local volatility \( \sigma_T \) for the first interval \([T_0, T_1]\) so as to exactly match the smile of the S&P 500 index for maturity \( T_i \), then set correlations \( \rho_{S,v} \) and \( \rho_{S,W} \) to calibrate at best the S&P 500 smile for longer-dated maturities.

A natural question in the discrete context is: why model forward variances \( V_T^{T,T_{i+1}} \) instead of working with Vix futures \( F_i \) directly, writing:

\[
 F_i' = F_i f^T \left(x_i, T_i \right)
\]

since the \( F' \) are driftless as well? Consider for example the option to
enter at \( T_i \) into a VS contract of maturity \( T_{\text{ret}} \). The variance of maturity \( T_{\text{ret}} \) is the average of \( n \) monthly variances. At time \( t = T_i \), the first monthly variance is known, as it is given by \( V^{T_i\rightarrow T_{\text{ret}}} \). However, we do not have access to the other variances, since what we are modelling are the \( F_j \), which are expectations of square roots of forward variances. While forward variances are additive, forward volatilities are not.

**Calibration of Vix futures and their smiles – options on realised variance**

Here, we demonstrate the capabilities of the continuous version of our model. We test its calibration to the Vix market, then discuss the pricing of spot-starting and forward-starting options on realised variance.

**Calibration of Vix futures and their smiles.** We model the full set of instantaneous forward variances. A discrete forward variance over \([T_i, T_{\text{ret}}]\), such as the 30-day variance that underlies Vix futures, is given by:

\[
V^{T_i\rightarrow T_{\text{ret}}} = \frac{1}{T_{\text{ret}} - T_i} \int_{T_i}^{T_{\text{ret}}} \sigma^2(T, t) dt
\]

Because \( \sigma^2(T, t) \) is a smooth function of \( T \), \( V^{T_i\rightarrow T_{\text{ret}}} \) is efficiently evaluated by Gaussian quadrature.

The initial variance curve \( \xi^T \) is a basic underlier in our model, along with \( S \). It would be natural to generate \( \xi^T \) from the VS volatilities derived from the S&P 500 vanilla smile. For each Vix future expiring at \( T_i \), we then need to calibrate \( \gamma, \beta, \zeta \) so that the Vix future itself and the implied volatilities of its options are matched. Under which conditions is this possible?

**Consistency of Vix futures/options with S&P 500 VS variances.** The settlement value of the Vix future of maturity \( T_i \) is the VS volatility for maturity \( T_{\text{ret}} \):

\[
F_i = \sqrt{V^{T_i\rightarrow T_{\text{ret}}} \cdot T_i}
\]

Using the Vix future \( F_i \), along with its calls and puts, one is able to replicate any European-style option on \( F_i \), in particular its square \( F_i^2 \). The consistency condition relating to the S&P 500 forward variance \( V^{T_i\rightarrow T_{\text{ret}}} \) and the Vix market is thus:

\[
V^{T_i\rightarrow T_{\text{ret}}} = F_i^2 + 2 \int_0^{F_i} \sigma^2(T, F_i) dF_i + 2 \int_0^{F_i} \sigma^2(T, F_i) dK + 2 \int_0^{F_i} C(T, F_i) dK
\]

(13)

where \( P_i(C_i) \) is the undiscounted price of a put (call) option on \( F_i \) of maturity \( T_i \). Practically, strikes used in the replication do not range from zero to infinity: we discard extreme strikes, which contribute little as Vix smiles are not very steep.

In our experience, the consistency condition above is not always met. Frequently, values of \( V^{T_i\rightarrow T_{\text{ret}}} \) derived through replication lie above the values we get from the S&P 500 VS market, meaning the Vix futures are too high, typically 0.5 of a volatility point. This is not always easily arbitrageable, as: one incurs bid/offer costs in trading variance swaps and Vix options; as is well known to practitioners, the settlement values of Vix futures, which involve market prices of traded S&P 500 options and are very sensitive to quotes of out-of-the-money puts often do not match one-month VS volatilities; and the liquidity of Vix futures and options decays quickly with their maturity.

In what follows, for the purpose of calibrating Vix futures and options, we use the variance curve \( \xi^T \) generated by the replication above rather than that given by the S&P 500 smile. Doing otherwise would be akin to trying to calibrate option prices using an incorrect value for the underlying.

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**An example of calibration.** We show results of a calibration done on March 18, 2008, for Vix futures and options of maturities April to August. Note that the liquidity of Vix futures and options dies off quickly after the first two or three listed maturities. Also, strikes that could be considered liquid, given the market conditions in March 2008, ranged from 20–35%. Calibration of triplets \((\gamma, \beta, \zeta)\) is done by least-squares minimisation. Figure 1 shows Vix futures and figure 2 Vix implied volatilities.

In figure 2, the implied volatility curves are stacked with respect to their maturities. The higher volatilities correspond to the April maturity, the lowest to the August maturity. While calibration is not perfect, it is very acceptable.

Had we used the discrete version of our model, calibration would be perfect by construction. Market and model curves would overlap exactly.

Table A shows the values we use for parameters \( \nu, \theta, k, k_1 \), and \( \rho \). Table B shows the values of \( \gamma, \beta, \zeta \) generated by calibration. Parameters \( \nu, \theta, k, k_1 \) and \( \rho \) control the correlation structure and volatilities of forward volatilities. Multiplying \( \nu \) and dividing \( \gamma \) by the same factor leaves the dynamics unchanged. We have chosen \( \nu \) so that the \( \zeta \) lie around one. As table B shows, \( \zeta \) does not vary much across maturities. This is desirable on the grounds of time-homogeneity.\(^6\) Note that for some maturities \( \beta \) vanishes. \( \xi^T \) is then a shifted lognormal.

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\(^5\) These are (90–95)%, or even after taking into account the difference in the definitions of VS volatilities and Vix volatilities.

\(^6\) Standard VS contracts define realised annualised variance on the sum of daily squared returns multiplied by factor 252/N where N is the number of observed returns, instead of dividing the sum of squared returns by the maturity of the VS contract.

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\(^6\) This is generally the case and is not specific to Vix smiles of March 18, 2008.
 mostly matters for shorter maturities and would stand as the sole component of the option’s value if VS volatilities were frozen. Here our focus is on the contribution of volatility of volatility. Let us assume no smile for forward variances (\( \gamma = 0, \zeta = 1 \)), use values for \( \nu, \theta, k, k_2, \) and \( \rho \) in Table A and consider the instantaneous volatility of a VS volatility as a function of its maturity, assuming the initial term structure of VS volatilities is flat \( (\xi^0 = \xi_0 T) \). The instantaneous volatility of volatility \( \sigma_{\nu \nu}^0 \) of \( \sqrt{V^0_T} \) is then given by:

$$
\sigma_{\nu \nu}^i = \nu \alpha_0 \left[ \beta (1 - e^{-k_1 T})^2 + (1 - \theta)^2 \right] \left( 1 - e^{-k_1 T} \right)^2 + 2 \rho \theta (1 - \theta) \left( 1 - e^{-k_2 T} \right) \left( 1 - e^{-k_2 T} \right) \right] \right]^{1/2}
$$

(15)

\( \sigma_{\nu \nu}^0 \) is plotted in Figure 3 for the set of parameters in Table A along with two other curves obtained using other sets of parameters. The unit for \( T \) is one month. Table C shows the corresponding values of \( \nu, \theta, k_1, k_2, \) and \( \rho \). We have chosen three values for \( \rho \), 0%, 90% and –70%, and have set the remaining parameters so that the term structures of instantaneous volatility of VS volatility are almost identical.

Table D shows prices for at-the-money options on realised variance for maturities of six months and one year, for an initial flat term structure of VS volatilities equal to 20%. In addition, Table D lists prices of: an option on forward realised variance (the realised variance is sampled over an interval of six months starting in six months); and a VS swaption (the option to enter in six months a VS contract of six months maturity struck at today’s level of forward volatility).

While prices of options on realised variance starting today are practically identical, prices for forward-starting options are different. This highlights the fact that while a term structure of volatility of volatility such as those shown in Figure 3 is all one needs to price spot-starting options on realised variance, it does not uniquely determine volatilities of forward volatilities. These depend on the correlation structure of forward volatilities. As an illustration, we plot in Figure 4 the instantaneous correlation at \( t = 0 \) of \( \sqrt{V^0_T} \) with:

$$
\sqrt{V^0_T} = (1 - \Delta \beta + 1) A
$$

for \( \Delta = 1 \) month and \( \beta = 0 \) to 11, for the three sets of parameters.

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A. Values for \( \nu, \theta, k_1, k_2, \) and \( \rho \) used in calibrating Vix smiles

| \( \nu \)  | 1300% |
| \( \theta \)  | 28%  |
| \( k_1 \)  | 8.0  |
| \( k_2 \)  | 0.35 |
| \( \rho \)  | 0%  |

B. Values of \( \gamma, \beta \) and \( \zeta \) calibrated from Vix smiles

| \( \gamma \)  | \( \beta \)  | \( \zeta \)  |
| Apr 16, 2008 | 8.7% | 21% | 100% |
| May 21, 2008 | 35% | 11% | 94% |
| Jun 18, 2008 | 35% | 11% | 96% |
| Jul 16, 2008 | 30% | 0% | 99% |
| Aug 20, 2008 | 2.4% | 0% | 100% |

C. The three sets of parameters used to generate curves in figure 3

<table>
<thead>
<tr>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu )</td>
<td>110%</td>
<td>113%</td>
</tr>
<tr>
<td>( \theta )</td>
<td>29%</td>
<td>29%</td>
</tr>
<tr>
<td>( k_1 )</td>
<td>8.0</td>
<td>12.0</td>
</tr>
<tr>
<td>( k_2 )</td>
<td>0.35</td>
<td>0.30</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0%</td>
<td>90%</td>
</tr>
</tbody>
</table>

D. Prices of at-the-money call options on spot-starting realised variance, forward realised variance and at-the-money swaptions

<table>
<thead>
<tr>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot-starting realised – six months</td>
<td>2.03%</td>
<td>2.03%</td>
</tr>
<tr>
<td>Spot-starting realised – one year</td>
<td>2.22%</td>
<td>2.21%</td>
</tr>
<tr>
<td>Six months realised in six months</td>
<td>2.04%</td>
<td>2.09%</td>
</tr>
<tr>
<td>Six months in six months swaption</td>
<td>2.28%</td>
<td>2.10%</td>
</tr>
</tbody>
</table>
In set 3, correlations of forward VS volatilities are much lower than in sets 1 and 2. For a given term structure of volatility of spot-starting volatility, the volatility of forward volatilities will be higher in set 3, explaining why prices of options on forward realised variance in table D increase when going from set 2 (p = 90%) through set 1 (p = 0%) to set 3 (p = -70%).

The impact on pricing would be even larger for more complex structures such as options on the spread of forward variances, which would presumably require a larger number of driving factors. Note that one-factor stochastic volatility models impose 100% instantaneous correlation between all forward variances.

Table D confirms the intuition that an option on realised forward variance should be more expensive than a variance swaption with the same forward date, maturity and strike, the difference in price being contributed by the randomness with which the underlying realises until maturity the variance that was observed at the forward date.

The smile of realised variance. Just as it is natural to incorporate information about the smiles of the basket’s underlyings in a basket option’s price, it is also natural to use information embedded in Vix smiles to price options on realised variance on the S&P 500, which can be thought of as options on a basket of forward variances, with part of their value coming from the kurtosis of daily returns.

Figure 5 shows the implied smile of realised variance for the S&P 500 using the values in table B, which were calibrated on Vix market smiles on March 18, for an option of six months’ maturity, using an initial flat VS term structure. This is the implied volatility of the underlying S in equation (14), assuming it has zero drift. The x-axis represents the moneyness of the volatility: \( K / \hat{S} \). Prices for the option on realised variance have been produced by simulating the spot process according to equation (10).

The smile in figure 5 should be taken with a pinch of salt. We know for example that equity market skews for indexes cannot be recovered by pricing them in a multi-asset local volatility framework with constant correlations between the underlying stocks – an issue known as correlation smile.

The vanilla smile. The dynamics of the variance curve not only affects the pricing of options on realised or implied variance but also determines the shape of vanilla options generated by the model. Let us derive an approximate expression for the at-the-money-forward skew at order one in the volatility of volatility using the same procedure as in Bergomi (2005), based on a calculation of the skewness of \( \ln S \), at order one in the volatility of volatility. At first order in the skewness \( S_x \) of \( \ln S \), the at-the-money-forward skew is given by Backus, Foresi & Wu (1997):

\[
\frac{d \hat{S}}{d \ln K} = \frac{S_x}{\sigma \sqrt{T}}
\]

Let us here consider the case of a flat initial term structure of variances and a lognormal dynamics:

\[
f_T(x, t) = e^{\frac{-\sigma^2}{2} t} \Phi(x) dt
\]

In line with our parameterisation, we write \( \sigma = 2\nu \), where \( \nu \) is the volatility of a very short VS volatility. At first order in the volatility of volatility \( \nu \) we get for the at-the-money-forward skew:

\[
\frac{d \hat{S}}{d \ln K} = \nu \chi \left[ \frac{k_t - (1 - e^{-k_T})}{k_T} + \theta \frac{k_t - (1 - e^{-k_T})}{k_T^2} \right]
\]

This expression is recovered from equation (8) in Bergomi (2005) by taking the limit \( \Delta \to 0 \), \( N = T/\Delta \) in the definition of \( \zeta(x, n) \).

In the limit \( T \to \infty \), the spot/volatility correlation function tends to zero, so \( \hat{S} \) decays like \( 1/\sqrt{T} \) and the skew decays as \( 1/T \), which is what we expect for independent returns.

For \( T \to 0 \), the skew does not vanish and tends to a finite value given in our model by:

\[
\frac{\nu \chi \left[ \frac{k_t - (1 - e^{-k_T})}{k_T} + \theta \frac{k_t - (1 - e^{-k_T})}{k_T^2} \right]}{2}
\]

This is common to all stochastic volatility models: the density they generate for \( \ln S \) becomes Gaussian for \( T \to 0 \). However, because \( \hat{S} \) vanishes like \( \sqrt{T} \), the skew, which is proportional to \( \hat{S} / \sqrt{T} \), tends to a finite value. This expression for the skew when \( T \to 0 \) agrees with the general result for stochastic volatility models:

\[
\frac{d \hat{S}}{d \ln K} = \frac{1}{4d \hat{S}} \left[ \frac{dS}{S \sqrt{V}} \right] \left[ \frac{dV}{V} \right]
\]

where \( V \) is the instantaneous variance.

We now check the accuracy of the approximate expression for the skew in equation (16) against its actual value. Figure 6 shows: the actual skew measured as the difference of the implied volatilities of the 95% and 105% strikes; and:
Notice how parameters in set 1 cause the skewness for the longer maturities in the figure to be flat, thus making the long-term skew acceptable quality. Raising the volatility of volatility or the level of spot/volatility correlations will cause it to deteriorate.

It is instructive to look at the dependence of the skewness

\[ \text{ST} \]

as a function of maturity.

As mentioned above, as \( T \) grows \( S_T \) will eventually decay like \( 1/VT \), which is typical of equity market smiles.

As given by equation (16). We can see that, while skews in figure 6 are not particularly large, \( S_T \), a dimensionless number, is of order one. Notice how parameters in set 1 cause the skewness for the longer maturities in the figure to be flat, thus making the long-term skew decay like \( 1/VT \), which is typical of equity market smiles.

For maturities up to two years, we have used the parameters in set 1 (see table C) and the following spot/volatility correlations: \( \rho_{\text{v}} = -70\% \), \( \rho_{\text{s}} = -35.7\% \). Even though it slightly overestimates the at-the-money-forward skew, approximation (16) is of acceptable quality. Raising the volatility of volatility or the level of spot/volatility correlations will cause it to deteriorate.

It is instructive to look at the dependence of the skewness

\[ ST \]

as a function of maturity.

Equation (16) highlights the dependence of the skew on correlations \( \rho_{\text{vx}} \) and \( \rho_{\text{vy}} \). Once values for parameters \( v, \theta, k_1, k_2 \) and \( \rho \) have been chosen, \( \rho_{\text{vx}} \) and \( \rho_{\text{vy}} \) can be used to control the overall level of the vanilla skew. However, as is made manifest in expressions (16) and (15), the decay of the skew and the term structure of volatilities of volatilities both depend on \( k_1 \) and \( k_2 \) as \( k_1, k_2 \) control both the volatility/volatility and spot return/volatility correlation functions. It will thus be difficult to have separate handles on the term structure of the vanilla skew and the term structure of volatility of volatility, at least in a two-factor framework.

**Conclusion**

In this article, we propose a model driven by easy-to-simulate Ornstein-Uhlenbeck processes that, in addition to providing control of the term structure of volatilities of volatilities and the correlation structure of forward volatilities, allows us to control the smile of forward volatilities, while remaining Markovian. We illustrate the impact of the correlation structure of forward volatilities on the pricing of options involving implied or realised forward variances.

We demonstrate how the model can be calibrated to Vix futures and options – exactly in the discrete version of the model – and use Vix market information to price options on the realised variance of the S&P 500 index. As liquidity of Vix futures and options expands to longer-dated maturities, and similar volatility derivatives on other indexes such as the Vstoxx gain popularity, we can expect an active market for payouts combining forward variances and the underlying index to take shape, requiring adequate modelling of the joint dynamics of the spot and its forward variances.

Finally, as the diversity of traded instruments grows, it is tempting to try to calibrate market prices of all available instruments, forcing on to the model’s parameters as much time-dependence as is needed to achieve this goal. We would like to caution against the excessive psychological benefit that perfect calibration may generate and the elation of push-button pricing. In our opinion, the issue of specifying the general dynamics of the model so that the carry levels for all gammas (be they spot, volatility or cross gamma) and forward skew are predictable and comfortable is much more relevant.

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