Pricing Asian Options

Lars B. Nielsen

February 2001
Abstract

This Master’s Thesis compares selected methods for pricing Asian options. Variance reduction methods in relation to Monte Carlo simulation are also studied. The market model assumed is a basic Black-Scholes model with a lognormal stock price and a deterministic interest rate. Asian options based on an arithmetic average are difficult to price since the distribution of the average is unknown. Consequently, numerical methods and various approximations must be applied.

The pricing methods considered are [Turnbull and Wakeman, 1991], [Levy, 1992], [Milevsky and Posner, 1998], [Vorst, 1992] and last but not least [Curran, 1994]. From these, two new methods are developed and one of them turns out to be surprisingly accurate. Nevertheless, numerical results show that the lower bound suggested by Curran is the most accurate and reliable method for a wide range of parameter values. It is also demonstrated that the price of the geometric Asian option is a very efficient control variate for Monte Carlo simulation of the arithmetic Asian option price.

Keywords: Asian option, average rate option, average price option, exotic option, path-dependent option, arithmetic average, geometric average, pricing bounds, option valuation, Edgeworth series expansion, Wilkinson approximation, conditional expectation, Monte Carlo simulation, control variate, Black-Scholes formula.
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>vii</td>
</tr>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Purpose of the thesis</td>
<td>2</td>
</tr>
<tr>
<td>2 Preliminaries</td>
<td>3</td>
</tr>
<tr>
<td>2.1 Options</td>
<td>3</td>
</tr>
<tr>
<td>2.2 Black-Scholes model</td>
<td>3</td>
</tr>
<tr>
<td>2.2.1 Black-Scholes option pricing formula</td>
<td>4</td>
</tr>
<tr>
<td>2.3 Arbitrage pricing</td>
<td>5</td>
</tr>
<tr>
<td>2.3.1 Fundamental PDE</td>
<td>5</td>
</tr>
<tr>
<td>2.3.2 Risk-neutral valuation</td>
<td>6</td>
</tr>
<tr>
<td>3 Asian options</td>
<td>7</td>
</tr>
<tr>
<td>3.1 Market model</td>
<td>7</td>
</tr>
<tr>
<td>3.2 Pricing</td>
<td>8</td>
</tr>
<tr>
<td>3.2.1 Averages</td>
<td>8</td>
</tr>
<tr>
<td>3.2.2 Risk-neutral valuation</td>
<td>9</td>
</tr>
<tr>
<td>4 Monte Carlo simulation</td>
<td>11</td>
</tr>
<tr>
<td>4.1 Standard Monte Carlo</td>
<td>11</td>
</tr>
<tr>
<td>4.1.1 Linear congruential generator</td>
<td>11</td>
</tr>
<tr>
<td>4.1.2 Inverse distribution function method</td>
<td>12</td>
</tr>
<tr>
<td>4.1.3 Box-Muller method</td>
<td>12</td>
</tr>
<tr>
<td>4.1.4 Sampling</td>
<td>13</td>
</tr>
<tr>
<td>4.1.5 Simulation</td>
<td>14</td>
</tr>
<tr>
<td>4.1.6 Method overview</td>
<td>16</td>
</tr>
<tr>
<td>4.2 Variance reduction techniques</td>
<td>16</td>
</tr>
<tr>
<td>4.2.1 Antithetic method</td>
<td>16</td>
</tr>
<tr>
<td>4.2.2 Control variates</td>
<td>18</td>
</tr>
<tr>
<td>4.3 Numerical results</td>
<td>20</td>
</tr>
<tr>
<td>Chapter</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td>4</td>
<td>4.3.1 Standard Monte Carlo</td>
</tr>
<tr>
<td>4</td>
<td>4.3.2 Antithetic method</td>
</tr>
<tr>
<td>4</td>
<td>4.3.3 Control variates</td>
</tr>
<tr>
<td>4</td>
<td>4.3.4 Conclusion</td>
</tr>
<tr>
<td>5</td>
<td>Turnbull and Wakeman algorithm</td>
</tr>
<tr>
<td>5</td>
<td>5.1 Generalized Edgeworth series expansion</td>
</tr>
<tr>
<td>5</td>
<td>5.2 Algorithm</td>
</tr>
<tr>
<td>5</td>
<td>5.2.1 Model</td>
</tr>
<tr>
<td>5</td>
<td>5.2.2 Moments and cumulants of the true distribution</td>
</tr>
<tr>
<td>5</td>
<td>5.2.3 Approximating distribution</td>
</tr>
<tr>
<td>5</td>
<td>5.2.4 Option price</td>
</tr>
<tr>
<td>5</td>
<td>5.2.5 Method overview</td>
</tr>
<tr>
<td>5</td>
<td>5.3 Numerical results</td>
</tr>
<tr>
<td>5</td>
<td>5.3.1 Conclusion</td>
</tr>
<tr>
<td>6</td>
<td>Milevsky and Posner method</td>
</tr>
<tr>
<td>6</td>
<td>6.1 Reciprocal gamma distribution</td>
</tr>
<tr>
<td>6</td>
<td>6.1.1 Gamma distribution</td>
</tr>
<tr>
<td>6</td>
<td>6.1.2 Reciprocal gamma distribution</td>
</tr>
<tr>
<td>6</td>
<td>6.2 Method</td>
</tr>
<tr>
<td>6</td>
<td>6.2.1 Model</td>
</tr>
<tr>
<td>6</td>
<td>6.2.2 Discrete averaging</td>
</tr>
<tr>
<td>6</td>
<td>6.2.3 Method overview</td>
</tr>
<tr>
<td>6</td>
<td>6.2.4 Continuous averaging</td>
</tr>
<tr>
<td>6</td>
<td>6.3 Numerical results</td>
</tr>
<tr>
<td>6</td>
<td>6.3.1 Conclusion</td>
</tr>
<tr>
<td>7</td>
<td>Vorst method</td>
</tr>
<tr>
<td>7</td>
<td>7.1 Method</td>
</tr>
<tr>
<td>7</td>
<td>7.1.1 Model</td>
</tr>
<tr>
<td>7</td>
<td>7.1.2 Geometric Asian option</td>
</tr>
<tr>
<td>7</td>
<td>7.1.3 Arithmetic option price and bounds</td>
</tr>
<tr>
<td>7</td>
<td>7.1.4 Method overview</td>
</tr>
<tr>
<td>7</td>
<td>7.2 Numerical results</td>
</tr>
<tr>
<td>7</td>
<td>7.2.1 Conclusion</td>
</tr>
<tr>
<td>8</td>
<td>Curran method</td>
</tr>
<tr>
<td>8</td>
<td>8.1 Method</td>
</tr>
<tr>
<td>8</td>
<td>8.1.1 Model</td>
</tr>
<tr>
<td>8</td>
<td>8.1.2 Bounds on the option price</td>
</tr>
</tbody>
</table>
CONTENTS

8.1.3 Method overview ........................................ 87
8.2 Numerical results ........................................... 88
  8.2.1 Conclusion ............................................... 88

9 Conclusion .................................................... 93

A Monte Carlo simulation ..................................... 97
  A.1 Visual Basic code ......................................... 97
    A.1.1 Standard Monte Carlo ................................ 97
    A.1.2 Antithetic method .................................... 100
    A.1.3 Control variates ...................................... 103

B Turnbull and Wakeman ....................................... 107
  B.1 Moments .................................................... 107
  B.2 Cumulants .................................................. 108
  B.3 Moment matching ......................................... 108
  B.4 Lognormal density and derivatives .................... 109
  B.5 Visual Basic code ........................................ 110

C Milevsky and Posner ......................................... 115
  C.1 Gamma distribution ....................................... 115
  C.2 Reciprocal gamma distribution ......................... 116
    C.2.1 Moments ............................................... 116
    C.2.2 Density and derivatives ............................. 117
  C.3 Moments .................................................... 119
  C.4 Visual Basic code ........................................ 121

D Vorst .......................................................... 127
  D.1 Visual Basic code ......................................... 127

E Curran .......................................................... 129
  E.1 Visual Basic code ......................................... 129
Preface

I have been fascinated by derivatives and in particular options ever since I first came across them. My interest only grew stronger, when I heard the news of the collapse of Barings Bank in 1995. Apparently, an employee in the bank’s Singapore division had lost the enormous amount of £850 million by unauthorized trading in the futures and options markets [Leeson, 1996]. In the early 1990s, huge losses were suffered by other derivatives market participants (e.g. Gibson Greetings, Procter and Gamble, Kidder Peabody and Orange County). These losses left the general impression that derivatives were very risky financial products. Though understandable, this is certainly not a fair characterization:

The stories behind the losses emphasize that derivatives can be used for hedging or speculation; that is, they can be used either to reduce risks or to take risks. The losses were experienced because derivatives were used inappropriately. People who had an implicit or explicit mandate to manage risks prudently decided to take big bets on the future direction of market variables. [Hull, 1997]

Here, at the beginning of a new decade, the interest in derivatives is rapidly increasing and the need for fast and accurate pricing and risk-measuring methods is ubiquitous. I have decided to focus on a small fraction of this ongoing task, namely the pricing of Asian options.

Guide to the chapters

Chapter 1 contains a brief introduction to Asian options followed by the purpose of this thesis. In Chapter 2 I take a look at the Black-Scholes model and present the famous Black-Scholes pricing formula for European call options. The concepts of arbitrage pricing and risk-neutral valuation are also introduced. Chapter 3 gives a more thorough treatment of Asian options and is the foundation of the remaining chapters. Chapter 4 reviews the basics of Monte
Carlo simulation and I study two variance reduction methods, namely the antithetic method and the control variate method. The pricing algorithm in [Turnbull and Wakeman, 1991] is the subject of Chapter 5. The distribution of the arithmetic average is approximated with a lognormal distribution by means of a generalized Edgeworth series expansion. This is a generalization of the method in [Levy, 1992]. In Chapter 6 I study the method suggested by [Milevsky and Posner, 1998], who use the reciprocal gamma distribution to approximate the average. I also combine the methods treated so far to form two new pricing methods. Chapter 7 presents the pricing methods in [Vorst, 1992], which cover both arithmetic and geometric Asian options. The geometric option is used to obtain bounds on the price of the arithmetic option. In Chapter 8 I look at the lower bound developed in [Curran, 1994] and I suggest how to combine this with an upper bound. The main conclusions of the thesis appear from Chapter 9.

**How to read this thesis**

In order to get the full benefit from the thesis a prior knowledge of mathematics and probability theory is implied. Readers already familiar with the Black-Scholes model and arbitrage pricing can skip Chapter 2 without loss of continuity. From time to time I refer to prices in US dollars but the conclusions apply to any currency. For technical reasons comma has been used as decimal point in the figures, I hope the reader will bear with this small inconsistency. Since it is easier to get a good grasp of multiple graphs when they are in color, all the figures are available at my website

- http://www.lbn.dk/master/
- http://home.imf.au.dk/larsbn/ (Mirror site)

Excel workbooks, Visual Basic code and this thesis in Adobe Portable Document Format (PDF) are also open to the public.

**Acknowledgments**

I would like to thank my advisor Jørgen Aase Nielsen for helpful suggestions regarding the choice of pricing methods. Allan Mortensen and Mikkel Svenstrup contributed by discussing specific issues with me for which I am grateful. I also owe my thanks to my family and my friends for their encouragement throughout the process. Finally, I am indebted to the woman in my life, Iben Delfs Sørensen, for her love and support.
Chapter 1

Introduction

Asian options are written on the average of the underlying asset over a pre-specified period. Consequently, they are also known as average rate or average price options. They were introduced in the oil market in the late 1970s and are now traded in the over-the-counter (OTC) markets all over the world. Asian options are path-dependent due to the averaging and they are often classified as exotic options even though the new generation of exotic options is structurally more complicated.

Compared to standard European options, Asian options have some obvious advantages. First of all, they are often cheaper and better suited for hedging purposes. It is not unusual for exporting companies to receive cash-flows in foreign exchange on a regular basis. Buying Asian put options on the exchange rate is a way to ensure that the company realizes at least the average exchange rate over the period in question. Secondly, Asian options reduce the risk of price manipulation near the maturity date, when the underlying is a thinly traded asset or commodity. Asian options are also seen in combination with corporate bonds, which allows the issuing company to share profits, if any, with its bondholders. For this to make sense, the profit of the company should of course depend on the underlying asset of the option.

The downside is that Asian options are hard to price. If the underlying asset is assumed to have a lognormal distribution as in the Black-Scholes model (see Chapter 2), then the arithmetic average does not have a known distribution, since the average is a sum of correlated lognormal random variables. This complicates the quest of a closed-form pricing expression for Asian options similar to the famous Black-Scholes pricing formula for European options.

We shall take a closer look at the Asian option in Chapter 3.
1.1 Purpose of the thesis

The purpose of this thesis is to:

1. Derive and implement selected methods for pricing Asian options.
2. Compare the accuracy of these methods using Monte Carlo simulation as yardstick.

More specifically it is my intention to:

• Study the use of Monte Carlo simulation including variance reduction methods in the context of pricing Asian options.
• Derive the different pricing formulae and all relevant parameters from the Black-Scholes model.
• Implement the methods in Visual Basic for Excel and calculate option prices and bounds for different values of key parameters.
• Compare the results with estimates obtained by means of Monte Carlo simulation to assess the accuracy of the methods.
• Make a comparison of the selected pricing methods with respect to accuracy and computational aspects.

In order to facilitate the implementation I have restrained the task as follows. The starting point is a basic Black-Scholes model with a lognormal stock price, a deterministic interest rate and an equivalent martingale measure. The averaging period is the entire lifetime of the option and the pricing takes place at the initial time. I only price call options since the price of the corresponding put option can be obtained from put-call parity (see e.g. [Bouaziz et al., 1994] or [Vorst, 1992]). Moreover, all options are of European type with a fixed strike and the average is non-weighted, which rule out premature redemption as well as flexible and floating strike Asian options.

I have selected the pricing methods based on their theoretical appeal, expected accuracy and practicability. It was an invariable demand that I was able to do all the programming myself and that the amount of time spent on the programming compared favorably with the benefit. I have excluded finite difference methods among others, because of the time consumption involved, not because of their lack of relevance. Recommendable partial differential equation (PDE) references in this context are [Dewyne and Wilmott, 1995a], [Dewyne and Wilmott, 1995b] and [Rogers and Shi, 1995].
Chapter 2

Preliminaries

Let us start off by taking a look at the famous Black-Scholes model and see how to price a European call option. This approach offers a good deal of intuition, which will come in handy later when we introduce the more complicated Asian option.

2.1 Options

A European option is an asset that gives the holder the right (but not the obligation) to buy or sell an underlying asset (e.g. a stock) at a predetermined date, called the maturity date or the expiration date. The term call option is used to indicate a right to buy, while put option indicates a right to sell. The strike price or the exercise price is the price at which you buy or sell the underlying asset at maturity. Thus, a European option is characterized by its strike price, maturity date, underlying asset and whether it is a call or a put option.

In contrast, American options can be exercised at more than one date, sometimes even continuously throughout the life of the option. As we shall see later, Asian options are options where the exercise price or the stock price at maturity is replaced by an average of the stock prices.

2.2 Black-Scholes model

Let the price of a stock $S_t$ follow the stochastic process

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$  \hspace{1cm} (2.1)
where $\mu$ and $\sigma$ are constants and $W$ is a standard Brownian motion. Likewise, there is a riskless bond with price process

$$dB_t = rB_t \, dt$$  \hfill (2.2)

where $r$ is a constant.

**Assumptions**

- There are no transaction costs or taxes.
- Short selling in unlimited amounts is allowed.
- All assets are divisible, i.e. you can buy or sell any fraction of the assets.
- The stock pays no dividends.
- There are no arbitrage opportunities.
- Trading takes place continuously.
- The riskless interest rate $r$ is constant and identical for all maturities.

The Black-Scholes model is often called ”lognormal”, because the stock price process is a Geometric Brownian Motion which implies that $S_t$ has a lognormal distribution (i.e. $\ln S_t$ has a normal distribution). The parameters $\mu$ and $\sigma$ can be interpreted as the ”instantaneous” expected rate of return and the ”instantaneous” standard deviation of the rate of return. We shall refer to $\mu$ as the drift and to $\sigma$ as the volatility.

### 2.2.1 Black-Scholes option pricing formula

In their now world famous article [Black and Scholes, 1973], Fischer Black and Myron Scholes gave the following pricing formula for a European call option.

**Theorem 1** The price $C(S_t, t)$ at time $t$ of a European call option on a stock $S_t$ with exercise price $K$ and maturity date $T$ is given by

$$C(S_t, t) = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2)$$  \hfill (2.3)
2.3. ARBITRAGE PRICING

where

\[ d_1 = \frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}} \]  \hspace{1cm} (2.4)

and

\[ d_2 = d_1 - \sigma \sqrt{T - t} \]  \hspace{1cm} (2.5)

\( \Phi \) is the standard normal distribution function. \( \sigma \) is the volatility of the stock and \( r \) is the riskless interest rate as defined in the Black-Scholes model above.

\textbf{Proof.} See [Black and Scholes, 1973] \hspace{1cm} \blacksquare

2.3 Arbitrage pricing

2.3.1 Fundamental PDE

The option pricing formula (2.3) is really the solution to a fundamental partial differential equation (PDE) for pricing derivatives. This PDE is deduced using a technique called arbitrage pricing. The basic idea is to replicate the derivative you want to price by trading continuously in the stock and bond. In the absence of arbitrage, the price of the derivative must be the same as the price of the portfolio of stock and bond.

\textbf{Theorem 2} The call option price \( C(S_t, t) \) satisfies the PDE

\[-rC(x, t) + C_t(x, t) + rxC_x(x, t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x, t) = 0, \]

\[ x \in [0, \infty[, \hspace{0.5cm} t \in [0, T[ \]  \hspace{1cm} (2.6)

with boundary condition

\[ c(x, T) = (x - K)^+, \hspace{0.5cm} x \in [0, \infty[ \]  \hspace{1cm} (2.7)

\( C_x \) and \( C_t \) are the first order derivatives with respect to the first and second arguments, while \( C_{xx} \) is the second order derivative with respect to the first argument.

\textbf{Proof.} See Chapter 6 in [Duffie, 1996]. \hspace{1cm} \blacksquare
2.3.2 Risk-neutral valuation

The Black-Scholes formula (2.3) can also be derived using risk-neutral valuation. Even though most investors are risk-averse, we can price derivatives as if all investors were risk-neutral. In their book Financial Calculus [Baxter and Rennie, 1996], Martin Baxter and Andrew Rennie give a brilliant exposition of risk-neutral valuation, including Ito’s lemma, equivalent martingale measures, The Martingale representation theorem and Girsanov’s theorem. Chapter 3 of the book is a must-read for novices in the area of continuous time finance.

**Theorem 3** The call option price $C(S_t, t)$ is given by 

$$C(S_t, t) = e^{-r(T-t)}E_Q[(S_T - K)^+]$$

where $Q$ is the equivalent martingale measure for the discounted stock $S_t/B_t$ and $E_Q$ is expectation under this measure.

**Proof.** See Chapter 3 in [Baxter and Rennie, 1996]. □
Chapter 3

Asian options

Asian options are options whose payoff depends on some kind of average of the underlying asset. Stock prices, stock indices, exchange rates, interest rates and commodities are all examples of underlying assets associated with Asian options traded in the over-the-counter market. For purely expositional reasons we assume that the underlying is a dividend-paying stock. The average can be specified in a number of ways and over various time periods, which gives rise to many different options within the Asian class.

3.1 Market model

In the rest of this thesis we shall use the Black-Scholes model of Chapter 2 and the underlying assumptions with a single exception. We assume that the stock pays a continuous dividend rate $q$, so the market consists of a risky stock $S$ and a riskless bond $B$, whose prices are given by the stochastic differential equations

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t$$
$$dB_t = rB_t dt$$

(3.1) (3.2)

$r$ is the riskless interest rate, $q$ is the stock dividend, $\sigma$ is the volatility of the stock price and $W_t$ is a standard Brownian Motion under an equivalent martingale measure $Q$. From now on all expectations are with respect to $Q$. According to [Karatzas and Shreve, 1991] the solutions to (3.1) and (3.2) are

$$S_t = S_0 \exp((r-q-\frac{1}{2}\sigma^2)t + \sigma W_t)$$
$$B_t = B_0 \exp(rt)$$

(3.3) (3.4)

From (3.3) we see that the stock price is lognormally distributed.
3.2 Pricing

3.2.1 Averages

As mentioned above, there are many ways to specify the average of the stock prices. Some examples are:

Discrete, arithmetic average

\[ A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i} \]  \hspace{1cm} (3.5)

Discrete, geometric average

\[ G = \left( \prod_{i=1}^{n} S_{t_i} \right)^\frac{1}{n} \]  \hspace{1cm} (3.6)

Continuous, arithmetic average

\[ A = \frac{1}{T - t} \int_{t}^{T} S_u \, du \]  \hspace{1cm} (3.7)

Continuous, geometric average

\[ G = \exp\left( \frac{1}{T - t} \int_{t}^{T} \ln S_u \, du \right) \]  \hspace{1cm} (3.8)

In the discrete averages there are \( n \) prices measured at the time points

\[ 0 \leq t_1 < t_2 < ... < t_n \leq T \]  \hspace{1cm} (3.9)

Usually, these time points are equidistant, e.g. weekly averaging over 1 year. If the average only consists of one price at time \( T \), then we have the European option which is a special case of the Asian option in this respect. The averaging period can be the entire lifetime of the option or some subset of it. In the latter case, the option is called forward-starting.

It is also possible to use a weighted average in which case the Asian option is referred to as flexible. We shall primarily look at the non-weighted, discrete, arithmetic average, but the geometric average plays an important role in some of the pricing methods. Since all traded Asian options are based on discrete averages, the continuous versions are only relevant in theoretical contexts.
3.2. PRICING

3.2.2 Risk-neutral valuation

Only call options are considered, so it is implicit in the rest of this thesis that an option is a call option. The payoff of a fixed strike Asian option with maturity date $T$ and strike price $K$ is the maximum of zero and the difference between the average $A$ and the strike price

$$\max(A - K, 0)$$

(3.10)

Using risk-neutral valuation, the arbitrage-free initial price of the option is

$$C = \exp(-rT)E[(A - K)^+]$$

(3.11)

where the expectation is with respect to the equivalent martingale measure $Q$ as mentioned above. In contrast, the floating strike Asian option has payoff

$$\max(S_T - A, 0)$$

(3.12)

so the initial price is

$$C = \exp(-rT)E[(S_T - A)^+]$$

(3.13)

We shall not discuss the floating strike option any further.

The price in (3.11) is not straightforward to calculate, since the distribution of the discrete arithmetic average is unknown. Each of the stock prices has a lognormal distribution, but unfortunately the sum of correlated lognormal random variables is not lognormally distributed. Thus, researchers have developed a number of different numerical methods to obtain the price. These include, but are not limited to, Monte Carlo simulation, approximations of the distribution of the arithmetic average, modifications to the geometric average and conditioning on it, binomial methods, numerical solutions to partial differential equations, Fourier transformation techniques and inversion of non-trivial Laplace transforms. We shall study some of these pricing methods in the following chapters.
CHAPTER 3. ASIAN OPTIONS
Chapter 4

Monte Carlo simulation

Monte Carlo simulation (MCS) is basically a method for evaluating integrals. Often, MCS is neither the best nor the fastest way to solve a given problem, but it is easy to implement and it is applicable to a wide range of problems. In our context, the input for the simulation algorithm is a set of random numbers. The literature on random numbers and random number generators (RNG) is large (see for instance [Glasserman, 1998]), and we shall only note that an ideal RNG produces a sequence of independent, identically distributed (iid.) numbers $u_1, u_2, \ldots$. In most applications, the uniform distribution on $[0, 1]$ is used.

4.1 Standard Monte Carlo

4.1.1 Linear congruential generator

Since true RNGs are difficult to find, a number of deterministic algorithms have been developed, so-called pseudo-random number generators. We shall use a linear congruential generator of the form

\begin{align*}
    x_n &= (ax_{n-1} + b) \mod c, \quad n = 1, 2, \ldots \quad (4.1) \\
    u_n &= \frac{x_n}{c}, \quad n = 1, 2, \ldots \quad (4.2)
\end{align*}

where the seed $x_0$ is chosen by the user and mod is the modulo function. The $u_n$’s are independent and $U(0, 1)$ distributed. Choosing good values for the parameters $a, b$ and $c$ can have a substantial effect on the outcome of the simulation. [Press et al., 1992] recommends

\begin{equation}
    a = 7^5, b = 0, c = 2^{31} - 1 \quad (4.3)
\end{equation}
c should be large since the algorithm can produce no more than c different numbers. A big advantage of this algorithm is that using the same seed \( x_0 \) leads to exactly the same numbers. This makes it useful for comparing results across different methods without storing the numbers.

### 4.1.2 Inverse distribution function method

We have to draw samples from the (log)normal distribution to simulate the stock prices. Since our random numbers are uniformly distributed we have to somehow transform them into the normal distribution. A very general method for this purpose is the inverse distribution function method. It pairs each of the \( U(0,1) \) numbers with the corresponding fractile for the \( N(0,1) \) distribution. E.g. 0.5 is transformed into 0 since

\[
\Phi^{-1}(0.5) = 0 \quad (4.4)
\]

and 0.975 is transformed into 1.96 since

\[
\Phi^{-1}(0.975) \approx 1.96 \quad (4.5)
\]

\( \Phi^{-1} \) is the inverse of the standard normal distribution function \( \Phi \)

\[
\Phi^{-1}(x) = \{ y \mid \Phi(y) = x \} \quad (4.6)
\]

The number \( \Phi^{-1}(x) \) is called the \( x \)-fractile of the standard normal distribution.

**Claim 4** The numbers transformed by the inverse distribution function method are standard normally distributed.

**Proof.** Let \( U \sim U(0,1) \). Then \( \Phi^{-1}(U) \sim N(0,1) \) since

\[
P(\Phi^{-1}(U) \leq x) = P(\Phi(\Phi^{-1}(U)) \leq \Phi(x)) = P(U \leq \Phi(x)) = \Phi(x)
\]

\[\blacksquare\]

The function \( \Phi^{-1} \) can be approximated very closely with a rational function which is convenient for programming. Alternatively, many computer software packages include tools for calculating these values.
4.1. STANDARD MONTE CARLO

4.1.3 Box-Muller method

Box and Muller have proposed a faster method (see [Johnson et al., 1994]) which has been used in our calculations. Let \( U_1 \) and \( U_2 \) be independent and uniformly distributed over \([0, 1]\). Then

\[
X_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2) \quad (4.7)
\]

\[
X_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2) \quad (4.8)
\]

are independent and standard normally distributed.

To sum up, our procedure so far is to generate an iid. sequence of \( U(0, 1) \) numbers using the linear congruential generator, and then transform the numbers pairwise to the \( N(0, 1) \) distribution using the Box-Muller method.

4.1.4 Sampling

Now, we are ready to sample! We know that the solution to

\[
dS_t = (r - q)S_t dt + \sigma S_t dW_t \quad (4.9)
\]

is

\[
S_t = S_0 \exp((r - q - \frac{1}{2} \sigma^2)t + \sigma W_t) \quad (4.10)
\]

and that

\[
W_t \sim N(0, t) \quad (4.11)
\]

so

\[
\ln \frac{S_t}{S_0} \sim N((r - q - \frac{1}{2} \sigma^2)t, \sigma^2 t) \quad (4.12)
\]

Samples \( S_t^{(i)} \) from the distribution of \( S_t \) are then obtained by setting

\[
S_t^{(i)} = S_0 \exp((r - q - \frac{1}{2} \sigma^2)t + \sigma \sqrt{t} \ Z^i), \quad i = 1, 2, ..., n \quad (4.13)
\]

where \( Z^i \sim N(0, 1), \quad i = 1, 2, ..., n, \) \( \) independent.

If we need to sample at different time points, i.e. a path, we can advance the solution as follows

\[
S_{t_j}^{(i)} = S_{t_{j-1}}^{(i)} \exp((r - q - \frac{1}{2} \sigma^2)(t_j - t_{j-1}) + \sigma \sqrt{t_j - t_{j-1}} \ Z^i, \quad j = 1, 2, ..., m, \quad i = 1, 2, ..., n \quad (4.14)
\]

where \( 0 < t_1 < t_2 < ... < t_m = T \). These samples are exact, i.e. there is no approximation error.
4.1.5 Simulation

The strong law of large numbers (see [Hoffmann-Jørgensen, 1994a]) states that

\[ \hat{S}_n = \frac{1}{n} \sum_{i=1}^{n} S_i \longrightarrow E[S_1] \quad a.s. \] (4.15)

This is really the heart of Monte Carlo simulation. We approximate the expectation of some random variable by sampling from its distribution and then averaging across replications (samples). \( \hat{S}_n \) as defined above is the Monte Carlo estimator of \( E[S_1] \). In this simple case, there is no need for MCS since we already know the mean of \( S_1 \). We do need MCS in the more general case where we want to evaluate \( E[f(S_t)] \) for some function \( f \) with the distribution of \( f(S_t) \) unknown.

Let \( X = (S_{t_1}, S_{t_2}, \ldots, S_{t_m}) \). We would like to evaluate \( E[f(X)] \) by Monte Carlo simulation for some continuous function \( f : \mathbb{R}^m \rightarrow \mathbb{R} \). Let \( Y = f(X) \) and let \( Y^{(i)} \) be the simulated values of \( Y \) obtained by sampling from the distributions of \( S_{t_j} \)

\[ Y^{(i)} = f(S_{t_1}^{(i)}, S_{t_2}^{(i)}, \ldots, S_{t_m}^{(i)}), \quad i = 1, 2, \ldots, n \] (4.16)

The Monte Carlo estimator of \( E[Y] \) is

\[ \hat{Y} = \frac{1}{n} \sum_{i=1}^{n} Y^{(i)} \] (4.17)

which, by the strong law of large numbers, is converging to the correct mean value

\[ \hat{Y} \rightarrow E[Y] \quad a.s. \quad as \; n \rightarrow \infty \] (4.18)

In other words, the estimator is unbiased. Moreover, the variance of the estimator is

\[ Var(\hat{Y}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} Y^{(i)}\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} Y^{(i)}\right) = \frac{Var(Y)}{n} \] (4.19)

These are two characteristics of a good estimator - convergence towards the correct mean and a variance which decreases as we use more samples.

Now that we have got an estimated price, a natural question would be: "How accurate is it?". We would like to calculate a price interval which contains the true price with an acceptable degree of certainty. The statistical
4.1. **STANDARD MONTE CARLO**

15

colorful concepts of significance level and confidence interval can help us here. The significance level $\alpha$ is the borderline probability, which separates events that are significant and non-significant. The implicit assumption is that events with probability less than $\alpha$ do not occur, i.e. they are not significant. In this thesis, we always use $\alpha = 5\%$.

A $(1 - \alpha)$ confidence interval is simply an interval which contains the true value behind the estimation with probability $(1 - \alpha)$. By the central limit theorem (see [Hoffmann-Jörgensen, 1994a])

$$
\frac{\hat{Y} - E[Y]}{\sigma_Y / \sqrt{n}} \rightarrow N(0, 1) \text{ in distribution} \quad (4.20)
$$

where $\sigma_Y$ is the standard deviation of $Y^{(1)}$, i.e. $\sigma_Y^2 = Var(Y^{(1)}) = Var(Y)$.

Since the sample variance $s^2_Y$ of $Y^{(1)}, Y^{(2)}, ..., Y^{(n)}$

$$
s^2_Y = \frac{1}{n-1} \sum_{i=1}^{n} (Y^{(i)} - \hat{Y})^2 \quad (4.21)
$$

is converging to $\sigma_Y^2$, we can replace $\sigma_Y$ with $s_Y$

$$
\frac{\hat{Y} - E[Y]}{s_Y / \sqrt{n}} \rightarrow N(0, 1) \text{ in distribution} \quad (4.22)
$$

**Proposition 5** A $(1 - \alpha)$ confidence interval for $E[Y]$ is

$$
[\hat{Y} - z_{1-\alpha/2} \frac{s_Y}{\sqrt{n}}, \hat{Y} + z_{1-\alpha/2} \frac{s_Y}{\sqrt{n}}] \quad (4.23)
$$

where $z_{\alpha}$ is the $\alpha$-fractile of the standard normal distribution.

**Proof.** From (4.22) we get

$$
1 - \alpha = P(z_{\alpha/2} \leq \frac{\hat{Y} - E[Y]}{s_Y / \sqrt{n}} \leq z_{1-\alpha/2})
= P(z_{\alpha/2} \frac{s_Y}{\sqrt{n}} - \hat{Y} \leq -E[Y] \leq z_{1-\alpha/2} \frac{s_Y}{\sqrt{n}} - \hat{Y})
= P(\hat{Y} - z_{1-\alpha/2} \frac{s_Y}{\sqrt{n}} \leq E[Y] \leq \hat{Y} - z_{\alpha/2} \frac{s_Y}{\sqrt{n}})
= P(\hat{Y} - z_{1-\alpha/2} \frac{s_Y}{\sqrt{n}} \leq E[Y] \leq \hat{Y} + z_{1-\alpha/2} \frac{s_Y}{\sqrt{n}})
$$

The last equation follows from the symmetry of the fractiles $z_{1-\alpha/2} = -z_{\alpha/2}$.

We conclude that the true mean value $E[Y]$ is in the above confidence interval.
with probability $1 - \alpha$. The confidence interval is asymptotically valid as $n \to \infty$, which means that it holds for large values of $n$. ■

A 95% confidence interval is then

$$[\hat{Y} - 1.96 \frac{s_Y}{\sqrt{n}}, \hat{Y} + 1.96 \frac{s_Y}{\sqrt{n}}]$$

(4.24)

When reporting estimates, one often includes the sample standard deviation of the estimator $s_Y/\sqrt{n}$. Doubling this amount is a fast way to form the interval (4.24) "mentally", since $2s_Y/\sqrt{n}$ is approximately the halfwidth of the interval.

### 4.1.6 Method overview

- Generate uniform random numbers.
- Transform the numbers to the standard normal distribution.
- Sample from the distribution of the stock prices.
- Compute discounted Asian option payoffs.
- Obtain an estimate of the option price by averaging across replications.
- Compute the sample standard deviation of the estimator and use it to form a confidence interval.

We would like to shrink the confidence interval as much as possible. This can be done either by reducing the variance or by increasing the number of paths. The latter is very expensive in terms of computer processing time, so we have to look at some variance reduction techniques.

### 4.2 Variance reduction techniques

#### 4.2.1 Antithetic method

In practice, one often uses 10,000 or 100,000 paths to simulate a price. Depending on the quality of the random numbers, this could lead to results that are not sufficiently precise. Unwanted correlation between the random numbers can render the simulation outcome totally useless. For instance, one can imagine that a set of "bad" parameter values for the linear congruential generator will result in either larger or smaller numbers than expected. We could try to solve this problem by "balancing" each large number with a corresponding small number. This is the intuition behind the
4.2. VARIANCE REDUCTION TECHNIQUES

antithetic method. Our uniform numbers \( U_1, U_2, \ldots \) can be matched with \( 1 - U_1, 1 - U_2, \ldots \) which are also uniformly distributed over \([0, 1]\) and independent. For each path we simulate with \( U_1, U_2, \ldots, U_m \), we generate a second path with \( 1 - U_1, 1 - U_2, \ldots, 1 - U_m \).

Formally, let \( \theta_1 = g(U_1, U_2, \ldots, U_m) \) and \( \bar{\theta}_1 = g(1 - U_1, 1 - U_2, \ldots, 1 - U_m) \) where \( g \) represents the simulation algorithm. We use \( n \) pairs of iid. antithetic outcomes \( (\theta_i, \bar{\theta}_i) \quad i = 1, 2, \ldots, n \) and a natural unbiased estimator is

\[
\hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} (\theta_i + \bar{\theta}_i) \tag{4.25}
\]

Calculating the standard deviation is slightly more difficult than usual since \( \theta_i \) and \( \bar{\theta}_i \) are not independent. We define the average of an outcome pair

\[
\overline{\theta}_i = \frac{\theta_i + \bar{\theta}_i}{2}, \quad i = 1, 2, \ldots, n \tag{4.26}
\]

and we look at \( \hat{\theta} \) as the sample mean of the \( \overline{\theta}_i \)'s

\[
\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \overline{\theta}_i = \frac{1}{n} \sum_{i=1}^{n} \frac{\theta_i + \bar{\theta}_i}{2} \tag{4.27}
\]

in accordance with (4.25). The sample standard deviation \( s_\theta \) is now obtained from

\[
s_\theta^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\overline{\theta}_i - \hat{\theta})^2 \tag{4.28}
\]

and an asymptotically valid \((1 - \alpha)\) confidence interval is

\[
\hat{\theta} - z_{1-\alpha/2}\frac{s_\theta}{\sqrt{n}}, \quad \hat{\theta} + z_{1-\alpha/2}\frac{s_\theta}{\sqrt{n}} \tag{4.29}
\]

An obvious alternative to using antithetic variates is simply to double the number of paths in a standard Monte Carlo simulation. In terms of computational time, these are more or less equivalent since generating random numbers is much faster than sampling (both methods require \( 2nm \) samples). In order to see which method is better, we compare the variance of the antithetic method

\[
Var(\overline{\theta}_1) = Var(\frac{\theta_1 + \bar{\theta}_1}{2}) = \frac{1}{4} Var(\theta_1) + \frac{1}{4} Var(\bar{\theta}_1) + \frac{1}{2} Cov(\theta_1, \bar{\theta}_1)
\]

\[
= \frac{1}{2} Var(\theta_1) + \frac{1}{2} Cov(\theta_1, \bar{\theta}_1) \tag{4.30}
\]
with the variance of a non-antithetic pair

\[
Var\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{1}{4}Var(\theta_1) + \frac{1}{4}Var(\theta_2) + \frac{1}{2}Cov(\theta_1, \theta_2)
\]

\[
= \frac{1}{2}Var(\theta_1)
\]  \hspace{1cm} (4.31)

The conclusion is that if \(Cov(\theta_1, \bar{\theta}_1) < 0\) then the antithetic method is preferred. [Glasserman, 1998] proves that a sufficient condition for antithetic variables to reduce variance is that the simulation output is monotone in the random numbers. In our case, the option price is increasing in the standard normal random numbers. He also gives a useful tip which applies to sampling from the standard normal distribution. If

\[
Z = \Phi^{-1}(U), \quad U \sim U(0, 1)
\]

then \(Z \sim N(0, 1)\) as already shown and due to symmetry

\[
-Z = \Phi^{-1}(1 - U)
\]  \hspace{1cm} (4.33)

This means that a simulation with normal random numbers \(Z_1, Z_2, \ldots, Z_m\) can be expanded with antithetic paths obtained from \(-Z_1, -Z_2, \ldots, -Z_m\). This is applicable even if the \(Z_i\)'s are not generated from the \(U_i\)'s by the inverse distribution function method.

### 4.2.2 Control variates

When making complicated calculations by means of computers, it is always a good idea to perform a calculation manually in a simple setup and compare the result with the output from the programming. Taking this a step further inspires us to use some of the information "hidden" in our calculations so far to improve our estimates. We could, for instance, price a standard European call option along with the Asian option and compare the outcome with the theoretical price obtained from the Black-Scholes formula. If we observed significant errors, we could correct the estimates with this new information. This is basically the control variates approach to reducing variance and thereby improving the simulation results.

More formally, the control variate is a random variable which depends on the simulation input and has a high degree of correlation with the variable we want to estimate. Moreover, we must be able to calculate the expectation of the control variate.
4.2. VARIANCE REDUCTION TECHNIQUES

As before, let $X = (S_{t_1}, S_{t_2}, ..., S_{t_m})$ and $Y = f(X)$. We use the function $h$ to represent the calculation of the control variate $W$, so

$$W = h(X)$$  \hspace{1cm} (4.34)

The simulated values of $W$ are

$$W^{(i)} = h(S_{t_1}^{(i)}, S_{t_2}^{(i)}, ..., S_{t_m}^{(i)}), \quad i = 1, 2, ..., n$$  \hspace{1cm} (4.35)

The new estimator is

$$\hat{Y}_{cv} = \frac{1}{n} \sum_{i=1}^{n} Y^{(i)} - (W^{(i)} - E[W])$$

$$= \frac{1}{n} \sum_{i=1}^{n} Y^{(i)} - W^{(i)} + E[W]$$  \hspace{1cm} (4.36)

The strong law of large numbers tells us that the estimator is unbiased, and the central limit theorem gives us a $(1 - \alpha)$ confidence interval

$$[\hat{Y}_{cv} - z_{1-\alpha/2} \frac{s_{cv}}{\sqrt{n}}, \hat{Y}_{cv} + z_{1-\alpha/2} \frac{s_{cv}}{\sqrt{n}}]$$  \hspace{1cm} (4.37)

where $s_{cv}$ is the sample standard deviation obtained from

$$s_{cv}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y^{(i)} - W^{(i)} + E[W] - \hat{Y}_{cv})^2$$  \hspace{1cm} (4.38)

The method can be further improved by using the estimator

$$\hat{Y}_{cvi} = \frac{1}{n} \sum_{i=1}^{n} Y^{(i)} - \beta (W^{(i)} - E[W])$$  \hspace{1cm} (4.39)

for some $\beta$. The details can be found in [Glasserman, 1998]. The downside is that more paths are needed to approximate the optional $\beta$.

**Geometric option**

We mentioned earlier that the control variable must have a high degree of correlation with the variable we want to estimate. This is indeed the case when pricing Asian arithmetic options with geometric options as control variates. Furthermore, the geometric average is lognormally distributed, which
makes it possible to give a closed form expression for the geometric option price. The discrete geometric average $G$ is

$$G = \left( \prod_{j=1}^{m} S_{t_j} \right)^{1/m}$$  \hspace{1cm} (4.40)

so the price of the geometric option is according to Chapter 7

$$C_G = \exp(-rT)\left\{ \exp(\mu_G + \frac{1}{2}\sigma_G^2)\Phi(d_1) - K\Phi(d_2) \right\}$$  \hspace{1cm} (4.41)

with

$$\mu_G = \ln S_0 + (r - q - \frac{1}{2}\sigma^2)T + h$$  \hspace{1cm} (4.42)

$$\sigma_G^2 = \sigma^2 h \frac{(2n+1)(n+1)}{6n}$$  \hspace{1cm} (4.43)

and

$$d_1 = \frac{\mu_G - \ln K + \sigma_G^2}{\sigma_G}$$  \hspace{1cm} (4.44)

$$d_2 = d_1 - \sigma_G$$  \hspace{1cm} (4.45)

### 4.3 Numerical results

Here, we present the results of using Monte Carlo simulation to price Asian options. We also explain in detail how the different averages, prices etc. have been calculated, and the Visual Basic code behind it all appears from Appendix A. In addition to standard Monte Carlo simulation, we have applied the antithetic method as well as control variates to see which method performs best.

The price of the Asian option is

$$C = \exp(-rT)E[\max(A - K, 0)]$$  \hspace{1cm} (4.46)

where $A$ is the discrete arithmetic average

$$A = \frac{1}{m} \sum_{j=1}^{m} S_{t_j}$$  \hspace{1cm} (4.47)

We use equidistant time points, so

$$t_j = t_{j-1} + h, \quad h = \frac{T}{m}, \quad t_0 = 0, \quad j = 1, 2, ..., m$$  \hspace{1cm} (4.48)
4.3. NUMERICAL RESULTS

The sample averages are

\[ A^{(i)} = \frac{1}{m} \sum_{j=1}^{m} S_{i,j}^{(i)}, \quad i = 1, 2, ..., n \]  (4.49)

with \( S_{i,j}^{(i)} \) given by (4.14).

4.3.1 Standard Monte Carlo

Sample option prices are calculated as

\[ C^{(i)} = \exp(-rT) \max(A^{(i)} - K, 0), \quad i = 1, 2, ..., n \]  (4.50)

which leads us to the Monte Carlo estimator

\[ \hat{C} = \frac{1}{n} \sum_{i=1}^{n} C^{(i)} \]  (4.51)

The sample variance \( s_C^2 \) is

\[ s_C^2 = \frac{1}{n-1} \sum_{i=1}^{n} (C^{(i)} - \hat{C})^2 \]  (4.52)

and a 95% confidence interval is

\[ [\hat{C} - 1.96 \frac{s_C}{\sqrt{n}}, \hat{C} + 1.96 \frac{s_C}{\sqrt{n}}] \]  (4.53)

Table 4.1 and Table 4.2 contain option prices and standard deviations for different values of the stock price volatility \( \sigma \) and the maturity \( T \) of the option. The other parameters of interest have been fixed at

\[ K = 100, \quad m = 12, \quad n = 50000, \quad q = 0, \quad r = 0.05, \quad S_0 = 100 \]  (4.54)

which means that both the initial stock price and the strike price are 100, the number of prices in the average is 12 (corresponding to monthly averaging when \( T = 1 \)), the stock pays no dividends, the interest rate is 5% and we have simulated with 50,000 paths. The last three columns in the tables represent the simulated price of a European option, the standard deviation of the estimator and the theoretical Black-Scholes value. The European option has been included to give us an idea of the accuracy of the Monte Carlo estimates. If the simulated European prices are significantly different
from the theoretical prices, we should expect that the Asian prices are also imprecise.

The Monte Carlo estimator of the European option price is

$$\hat{C}_E = \frac{1}{n} \sum_{i=1}^{n} C_E^{(i)}$$

(4.55)

where the samples are

$$C_E^{(i)} = \exp(-rT) \max(S_{tm}^{(i)} - K, 0), \quad i = 1, 2, ..., n$$

(4.56)

The sample variance $s_E^2$ is

$$s^2_E = \frac{1}{n-1} \sum_{i=1}^{n} (C_E^{(i)} - \hat{C}_E)^2$$

(4.57)

and a 95% confidence interval is

$$[\hat{C}_E - 1.96 \frac{s_E}{\sqrt{n}}, \hat{C}_E + 1.96 \frac{s_E}{\sqrt{n}}]$$

(4.58)

The theoretical price $C_E$ is obtained from (2.3).

It is obvious from Table 4.1 that the standard deviations are too large to be of any good. As we would expect, both the option price and the standard deviation are increasing in the stock price volatility. This can be seen in Figure 4.1 where we also see that the spread between the European and Asian prices increase as volatility increases. The Asian price is lower than the European price mainly because the averaging tends to lower the variance of the terminal payoff, and hence lower the option price. Figure 4.1 also shows the endpoints of the 95% confidence interval for $\hat{C}_E$ as well as the Black-Scholes price. $\hat{C}_E$ has been subtracted from all values to make the graphs distinguishable from each other. The true price is inside the confidence interval, but the interval is so wide that we really can’t use our estimates to anything. As can be seen, the Monte Carlo simulation consistently underprices the European option, which could indicate that our random numbers are not as uncorrelated as we would wish.

Table 4.2 is similar to Table 4.1 but this time we have priced options with maturities from half a year up to five years. It is evident that both prices and standard deviations increase as maturity increases. This is also illustrated in Figure 4.2. The confidence intervals are too wide for our purpose, so we could consider increasing the number of paths. From (4.58) we see that doubling the precision requires four times as many paths. If we demand that the confidence interval is no wider than 0.01, i.e. 1 cent, then we need more than 250 million paths to price the option with $\sigma=0.50$ in Table 4.1. This would probably take forever on a standard PC, so variance reduction methods are indeed necessary.
4.3. NUMERICAL RESULTS

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$T$</th>
<th>$C$</th>
<th>$s_{C}/\sqrt{n}$</th>
<th>$C_E$</th>
<th>$s_{E}/\sqrt{n}$</th>
<th>$C_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>1</td>
<td>2.92</td>
<td>0.0114</td>
<td>5.28</td>
<td>0.0196</td>
<td>5.28</td>
</tr>
<tr>
<td>0.10</td>
<td>1</td>
<td>3.89</td>
<td>0.0200</td>
<td>6.80</td>
<td>0.0344</td>
<td>6.80</td>
</tr>
<tr>
<td>0.15</td>
<td>1</td>
<td>4.99</td>
<td>0.0287</td>
<td>8.59</td>
<td>0.0495</td>
<td>8.59</td>
</tr>
<tr>
<td>0.20</td>
<td>1</td>
<td>6.13</td>
<td>0.0377</td>
<td>10.44</td>
<td>0.0654</td>
<td>10.45</td>
</tr>
<tr>
<td>0.25</td>
<td>1</td>
<td>7.28</td>
<td>0.0471</td>
<td>12.32</td>
<td>0.0821</td>
<td>12.34</td>
</tr>
<tr>
<td>0.30</td>
<td>1</td>
<td>8.43</td>
<td>0.0568</td>
<td>14.20</td>
<td>0.0998</td>
<td>14.23</td>
</tr>
<tr>
<td>0.35</td>
<td>1</td>
<td>9.58</td>
<td>0.0670</td>
<td>16.09</td>
<td>0.1185</td>
<td>16.13</td>
</tr>
<tr>
<td>0.40</td>
<td>1</td>
<td>10.73</td>
<td>0.0776</td>
<td>17.97</td>
<td>0.1383</td>
<td>18.02</td>
</tr>
<tr>
<td>0.45</td>
<td>1</td>
<td>11.88</td>
<td>0.0886</td>
<td>19.84</td>
<td>0.1593</td>
<td>19.91</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>13.03</td>
<td>0.1001</td>
<td>21.70</td>
<td>0.1815</td>
<td>21.79</td>
</tr>
</tbody>
</table>

Table 4.1: Monte Carlo option prices for different volatilities. $K=100$, $m=12$, $n=50000$, $q=0$, $r=0.05$, $S_0=100$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$T$</th>
<th>$C$</th>
<th>$s_{C}/\sqrt{n}$</th>
<th>$C_E$</th>
<th>$s_{E}/\sqrt{n}$</th>
<th>$C_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.5</td>
<td>4.09</td>
<td>0.0256</td>
<td>6.88</td>
<td>0.0435</td>
<td>6.89</td>
</tr>
<tr>
<td>0.20</td>
<td>1.0</td>
<td>6.13</td>
<td>0.0377</td>
<td>10.44</td>
<td>0.0654</td>
<td>10.45</td>
</tr>
<tr>
<td>0.20</td>
<td>1.5</td>
<td>7.80</td>
<td>0.0476</td>
<td>13.43</td>
<td>0.0837</td>
<td>13.44</td>
</tr>
<tr>
<td>0.20</td>
<td>2.0</td>
<td>9.28</td>
<td>0.0561</td>
<td>16.10</td>
<td>0.1000</td>
<td>16.13</td>
</tr>
<tr>
<td>0.20</td>
<td>2.5</td>
<td>10.62</td>
<td>0.0638</td>
<td>18.57</td>
<td>0.1151</td>
<td>18.60</td>
</tr>
<tr>
<td>0.20</td>
<td>3.0</td>
<td>11.85</td>
<td>0.0708</td>
<td>20.89</td>
<td>0.1293</td>
<td>20.92</td>
</tr>
<tr>
<td>0.20</td>
<td>3.5</td>
<td>13.00</td>
<td>0.0773</td>
<td>23.08</td>
<td>0.1428</td>
<td>23.12</td>
</tr>
<tr>
<td>0.20</td>
<td>4.0</td>
<td>14.08</td>
<td>0.0835</td>
<td>25.17</td>
<td>0.1558</td>
<td>25.21</td>
</tr>
<tr>
<td>0.20</td>
<td>4.5</td>
<td>15.09</td>
<td>0.0892</td>
<td>27.16</td>
<td>0.1683</td>
<td>27.22</td>
</tr>
<tr>
<td>0.20</td>
<td>5.0</td>
<td>16.05</td>
<td>0.0947</td>
<td>29.08</td>
<td>0.1804</td>
<td>29.14</td>
</tr>
</tbody>
</table>

Table 4.2: Monte Carlo option prices for different maturities. $K=100$, $m=12$, $n=50000$, $q=0$, $r=0.05$, $S_0=100$. 
Figure 4.1: Top: Monte Carlo estimates for Asian and European option prices. Bottom: 95% confidence intervals and theoretical European option prices. All values subtracted by Monte Carlo estimate.
Figure 4.2: Top: Monte Carlo estimates for Asian and European option prices. Bottom: 95% confidence intervals and theoretical European option prices. All values subtracted by Monte Carlo estimate.
4.3.2 Antithetic method

For each \( C^{(i)} \) in (4.50) we generate a \( \tilde{C}^{(i)} \) by using \( \tilde{Z}^{i,j} = -Z^{i,j}, \; j = 1, 2, \ldots, m \) to simulate the stock prices. The Monte Carlo estimator is

\[
\hat{C} = \frac{1}{n} \sum_{i=1}^{n} \frac{C^{(i)} + \tilde{C}^{(i)}}{2}
\]

(4.59)

and a 95% confidence interval is

\[
[\hat{C} - 1.96 \frac{s_C}{\sqrt{n}}, \; \hat{C} + 1.96 \frac{s_C}{\sqrt{n}}]
\]

(4.60)

with

\[
s^2_C = \frac{1}{n-1} \sum_{i=1}^{n} \left( \frac{C^{(i)} + \tilde{C}^{(i)}}{2} - \hat{C} \right)^2
\]

(4.61)

The European option prices are simulated in a similar fashion.

Table 4.3 and Table 4.4 contain simulated option prices for different volatilities and maturities. The same remarks as for the standard method above apply. It is worth noting that the Asian prices have increased by up to 7 cents (see also Figure 4.3) whereas the European prices only have increased by up to 2 cents. Since the standard method underpriced the European options, it might have underpriced the Asian options as well, so we expect that the higher antithetic estimates of the Asian prices are better. Even though the antithetic method has reduced the standard deviations by a factor two, we are still not satisfied with the accuracy. We postpone our final conclusion regarding the antithetic method until we have seen how the other variance reduction method performs.

4.3.3 Control variates

In addition to (4.49), the sample geometric averages are

\[
G^{(i)} = \left( \prod_{j=1}^{m} S^{(i)}_{t_j} \right)^{1/m}, \quad i = 1, 2, \ldots, n
\]

(4.62)

and sample geometric option prices are calculated as

\[
C^{(i)}_G = \exp(-rT) \max(G^{(i)} - K, 0), \quad i = 1, 2, \ldots, n
\]

(4.63)
### 4.3. Numerical Results

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$T$</th>
<th>$C$</th>
<th>$s_{C}/\sqrt{n}$</th>
<th>$C_E$</th>
<th>$s_{E}/\sqrt{n}$</th>
<th>$C_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>1</td>
<td>2.93</td>
<td>0.0028</td>
<td>5.28</td>
<td>0.0044</td>
<td>5.28</td>
</tr>
<tr>
<td>0.10</td>
<td>1</td>
<td>3.90</td>
<td>0.0079</td>
<td>6.80</td>
<td>0.0133</td>
<td>6.80</td>
</tr>
<tr>
<td>0.15</td>
<td>1</td>
<td>5.01</td>
<td>0.0131</td>
<td>8.59</td>
<td>0.0226</td>
<td>8.59</td>
</tr>
<tr>
<td>0.20</td>
<td>1</td>
<td>6.15</td>
<td>0.0186</td>
<td>10.44</td>
<td>0.0325</td>
<td>10.45</td>
</tr>
<tr>
<td>0.25</td>
<td>1</td>
<td>7.31</td>
<td>0.0243</td>
<td>12.32</td>
<td>0.0431</td>
<td>12.34</td>
</tr>
<tr>
<td>0.30</td>
<td>1</td>
<td>8.47</td>
<td>0.0303</td>
<td>14.21</td>
<td>0.0545</td>
<td>14.23</td>
</tr>
<tr>
<td>0.35</td>
<td>1</td>
<td>9.63</td>
<td>0.0367</td>
<td>16.10</td>
<td>0.0666</td>
<td>16.13</td>
</tr>
<tr>
<td>0.40</td>
<td>1</td>
<td>10.79</td>
<td>0.0434</td>
<td>17.98</td>
<td>0.0797</td>
<td>18.02</td>
</tr>
<tr>
<td>0.45</td>
<td>1</td>
<td>11.95</td>
<td>0.0506</td>
<td>19.85</td>
<td>0.0936</td>
<td>19.91</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>13.10</td>
<td>0.0581</td>
<td>21.72</td>
<td>0.1086</td>
<td>21.79</td>
</tr>
</tbody>
</table>

Table 4.3: Monte Carlo option prices for different volatilities. Antithetic method. $K=100$, $m=12$, $n=50000$, $q=0$, $r=0.05$, $S_0=100$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$T$</th>
<th>$C$</th>
<th>$s_{C}/\sqrt{n}$</th>
<th>$C_E$</th>
<th>$s_{E}/\sqrt{n}$</th>
<th>$C_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.5</td>
<td>4.11</td>
<td>0.0127</td>
<td>6.88</td>
<td>0.0218</td>
<td>6.89</td>
</tr>
<tr>
<td>0.20</td>
<td>1.0</td>
<td>6.15</td>
<td>0.0186</td>
<td>10.44</td>
<td>0.0325</td>
<td>10.45</td>
</tr>
<tr>
<td>0.20</td>
<td>1.5</td>
<td>7.83</td>
<td>0.0232</td>
<td>13.43</td>
<td>0.0414</td>
<td>13.44</td>
</tr>
<tr>
<td>0.20</td>
<td>2.0</td>
<td>9.31</td>
<td>0.0272</td>
<td>16.11</td>
<td>0.0495</td>
<td>16.13</td>
</tr>
<tr>
<td>0.20</td>
<td>2.5</td>
<td>10.66</td>
<td>0.0307</td>
<td>18.58</td>
<td>0.0569</td>
<td>18.60</td>
</tr>
<tr>
<td>0.20</td>
<td>3.0</td>
<td>11.89</td>
<td>0.0340</td>
<td>20.89</td>
<td>0.0640</td>
<td>20.92</td>
</tr>
<tr>
<td>0.20</td>
<td>3.5</td>
<td>13.05</td>
<td>0.0370</td>
<td>23.08</td>
<td>0.0708</td>
<td>23.12</td>
</tr>
<tr>
<td>0.20</td>
<td>4.0</td>
<td>14.13</td>
<td>0.0399</td>
<td>25.17</td>
<td>0.0774</td>
<td>25.21</td>
</tr>
<tr>
<td>0.20</td>
<td>4.5</td>
<td>15.15</td>
<td>0.0426</td>
<td>27.16</td>
<td>0.0838</td>
<td>27.22</td>
</tr>
<tr>
<td>0.20</td>
<td>5.0</td>
<td>16.12</td>
<td>0.0452</td>
<td>29.08</td>
<td>0.0901</td>
<td>29.14</td>
</tr>
</tbody>
</table>

Table 4.4: Monte Carlo option prices for different maturities. Antithetic method. $K=100$, $m=12$, $n=50000$, $q=0$, $r=0.05$, $S_0=100$. 
Figure 4.3: Difference between Monte Carlo estimates with and without variance reduction.
4.3. NUMERICAL RESULTS

whereas the true geometric option price \( C_G \) is given by (4.41). The Monte Carlo estimator of the Asian option price is then

\[
\hat{C} = \frac{1}{n} \sum_{i=1}^{n} C^{(i)} - C^{(i)}_G + C_G
\]  

(4.64)

and a 95% confidence interval is

\[
[\hat{C} - 1.96 \frac{s_C}{\sqrt{n}}, \hat{C} + 1.96 \frac{s_C}{\sqrt{n}}]
\]  

(4.65)

with

\[
s^2_C = \frac{1}{n-1} \sum_{i=1}^{n} (C^{(i)} - C^{(i)}_G + C_G - \hat{C})^2
\]  

(4.66)

Likewise, the Monte Carlo estimator of the geometric option price is

\[
\hat{C}_G = \frac{1}{n} \sum_{i=1}^{n} C^{(i)}_G
\]  

(4.67)

The sample variance is

\[
s^2_G = \frac{1}{n-1} \sum_{i=1}^{n} (C^{(i)}_G - \hat{C}_G)^2
\]  

(4.68)

so a 95% confidence interval is

\[
[\hat{C}_G - 1.96 \frac{s_G}{\sqrt{n}}, \hat{C}_G + 1.96 \frac{s_G}{\sqrt{n}}]
\]  

(4.69)

Table 4.5 and Table 4.6 show simulated option prices for the Asian option and the geometric option as well as the theoretical geometric price. The prices are close for small volatilities and small maturities (see Figure 4.4 and Figure 4.5). The geometric options are cheaper than the Asian options because the geometric average is always lower than the arithmetic average. The simulation has consistently underpriced the geometric option, which again could indicate correlation between the random numbers. The standard deviations of \( \hat{C} \) have been reduced by a factor 9 or more which has dramatically narrowed the confidence intervals as seen in Figure 4.4 and Figure 4.5. From these it is also evident that the estimates from the antithetic method and the control variates method are almost the same.
<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>T</th>
<th>C</th>
<th>$s_C / \sqrt{n}$</th>
<th>$C_G$</th>
<th>$s_{C_G} / \sqrt{n}$</th>
<th>$\hat{C}_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>1</td>
<td>2.93</td>
<td>0.0001</td>
<td>2.89</td>
<td>0.0113</td>
<td>2.90</td>
</tr>
<tr>
<td>0.10</td>
<td>1</td>
<td>3.90</td>
<td>0.0004</td>
<td>3.82</td>
<td>0.0197</td>
<td>3.84</td>
</tr>
<tr>
<td>0.15</td>
<td>1</td>
<td>5.01</td>
<td>0.0009</td>
<td>4.86</td>
<td>0.0281</td>
<td>4.88</td>
</tr>
<tr>
<td>0.20</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>5.91</td>
<td>0.0366</td>
<td>5.94</td>
</tr>
<tr>
<td>0.25</td>
<td>1</td>
<td>7.31</td>
<td>0.0024</td>
<td>6.96</td>
<td>0.0453</td>
<td>6.99</td>
</tr>
<tr>
<td>0.30</td>
<td>1</td>
<td>8.47</td>
<td>0.0035</td>
<td>7.98</td>
<td>0.0542</td>
<td>8.02</td>
</tr>
<tr>
<td>0.35</td>
<td>1</td>
<td>9.64</td>
<td>0.0049</td>
<td>8.99</td>
<td>0.0632</td>
<td>9.04</td>
</tr>
<tr>
<td>0.40</td>
<td>1</td>
<td>10.80</td>
<td>0.0065</td>
<td>9.97</td>
<td>0.0725</td>
<td>10.03</td>
</tr>
<tr>
<td>0.45</td>
<td>1</td>
<td>11.96</td>
<td>0.0085</td>
<td>10.92</td>
<td>0.0820</td>
<td>11.00</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>13.11</td>
<td>0.0107</td>
<td>11.85</td>
<td>0.0917</td>
<td>11.94</td>
</tr>
</tbody>
</table>

Table 4.5: Monte Carlo option prices for different volatilities. Control variates. $K=100$, $m=12$, $n=50000$, $q=0$, $r=0.05$, $S_0=100$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>T</th>
<th>C</th>
<th>$s_C / \sqrt{n}$</th>
<th>$C_G$</th>
<th>$s_{C_G} / \sqrt{n}$</th>
<th>$\hat{C}_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.5</td>
<td>4.11</td>
<td>0.0007</td>
<td>3.99</td>
<td>0.0250</td>
<td>4.01</td>
</tr>
<tr>
<td>0.20</td>
<td>1.0</td>
<td>6.16</td>
<td>0.0016</td>
<td>5.91</td>
<td>0.0366</td>
<td>5.94</td>
</tr>
<tr>
<td>0.20</td>
<td>1.5</td>
<td>7.84</td>
<td>0.0025</td>
<td>7.47</td>
<td>0.0457</td>
<td>7.50</td>
</tr>
<tr>
<td>0.20</td>
<td>2.0</td>
<td>9.32</td>
<td>0.0034</td>
<td>8.81</td>
<td>0.0535</td>
<td>8.85</td>
</tr>
<tr>
<td>0.20</td>
<td>2.5</td>
<td>10.66</td>
<td>0.0045</td>
<td>10.01</td>
<td>0.0604</td>
<td>10.05</td>
</tr>
<tr>
<td>0.20</td>
<td>3.0</td>
<td>11.90</td>
<td>0.0055</td>
<td>11.10</td>
<td>0.0686</td>
<td>11.15</td>
</tr>
<tr>
<td>0.20</td>
<td>3.5</td>
<td>13.05</td>
<td>0.0066</td>
<td>12.10</td>
<td>0.0722</td>
<td>12.16</td>
</tr>
<tr>
<td>0.20</td>
<td>4.0</td>
<td>14.14</td>
<td>0.0078</td>
<td>13.03</td>
<td>0.0774</td>
<td>13.09</td>
</tr>
<tr>
<td>0.20</td>
<td>4.5</td>
<td>15.16</td>
<td>0.0090</td>
<td>13.90</td>
<td>0.0822</td>
<td>13.96</td>
</tr>
<tr>
<td>0.20</td>
<td>5.0</td>
<td>16.12</td>
<td>0.0102</td>
<td>14.71</td>
<td>0.0867</td>
<td>14.77</td>
</tr>
</tbody>
</table>

Table 4.6: Monte Carlo option prices for different maturities. Control variates. $K=100$, $m=12$, $n=50000$, $q=0$, $r=0.05$, $S_0=100$. 
Figure 4.4: Top: Monte Carlo option prices of Asian options with arithmetic and geometric averages. Control variates. Bottom: 95% confidence intervals (estimate subtracted) and difference between antithetic method and control variates.
Figure 4.5: Top: Monte Carlo option prices of Asian options with arithmetic and geometric averages. Control variates. Bottom: 95% confidence intervals (estimate subtracted) and difference between antithetic method and control variates.
4.3. NUMERICAL RESULTS

4.3.4 Conclusion

We have used Monte Carlo simulation to price various Asian and European options. From the simulations we conclude that the standard method is not accurate enough and that increasing the number of paths is not computationally feasible. This is in accordance with the results in [Kemna and Vorst, 1990]. The random numbers generated from the linear congruential generator and transformed by the Box-Muller method showed signs of correlation, so using more sophisticated methods to generate random numbers could prove advantageous. We applied two different variance reduction methods which both gave very good estimates, but the antithetic method did not reduce the standard deviations to an acceptable level. In contrast, using the Asian option based on a geometric average as a control variate for the arithmetic Asian option turned out extremely well due to the strong correlation between the two kinds of averages. We obtained 95% confidence intervals only a few cents wide even for large volatilities and large maturities.
Chapter 5

Turnbull and Wakeman algorithm

[Turnbull and Wakeman, 1991] price Asian options by approximating the distribution of the arithmetic average with a lognormal distribution. Similar methods have been used by [Levy, 1992] and others, but Turnbull and Wakeman use a generalized Edgeworth series expansion which also takes differences in skewness and kurtosis into account. This approach relies on the fact that all the moments of the arithmetic average can easily be calculated even though the distribution of the average is unknown. The algorithm is very fast and, as we shall see, it is relatively precise.

5.1 Generalized Edgeworth series expansion

The generalized Edgeworth series expansion is a technique for approximating a distribution function $F$ (the true distribution) with an alternative distribution function $G$ (the approximating distribution). It has similarities to the well-known Taylor series expansion of an analytical function. We assume that $F$ and $G$ have continuous density functions $f$ and $g$ and that the random variable $X$ has distribution $F$.

Define the $m$’th moment

$$
\alpha_m(F) = \int_{-\infty}^{\infty} x^m f(x) dx
= E[X^m], \quad m = 1, 2, ...
$$

(5.1)
and the $m$'th central moment

$$
\mu_m(F) = \int_{-\infty}^{\infty} (x - \alpha_1(F))^m f(x) \, dx
= E[(X - E[X])^m], \quad m = 1, 2, \ldots
$$

(5.2)

and the first four cumulants

$$
\begin{align*}
\kappa_1(F) &= \alpha_1(F) = E[X] \\
\kappa_2(F) &= \mu_2(F) = E[(X - E[X])^2] \\
\kappa_3(F) &= \mu_3(F) = E[(X - E[X])^3] \\
\kappa_4(F) &= \mu_4(F) - 3(\mu_2(F))^2 \\
&= E[(X - E[X])^4] - 3(E[(X - E[X])^2])^2
\end{align*}
$$

(5.3) (5.4) (5.5) (5.6)

$\kappa_1$ is the mean, $\kappa_2$ is the variance, $\kappa_3$ is a measure of skewness and $\kappa_4$ is a measure of kurtosis. We define $\alpha_m(G)$, $\mu_m(G)$ and $\kappa_m(G)$ likewise as above.

Theorem 6 Assume that the first five moments of both distributions exist and that $G$ is five times differentiable. We can then expand $f(x)$ in terms of $g(x)$ as follows

$$
\begin{align*}
f(x) &= g(x) + \frac{\kappa_2(F) - \kappa_2(G)}{2!} \frac{d^2 g}{dx^2}(x) - \frac{\kappa_3(F) - \kappa_3(G)}{3!} \frac{d^3 g}{dx^3}(x) \\
&\quad + \frac{\kappa_4(F) - \kappa_4(G) + 3(\kappa_2(F) - \kappa_2(G))^2}{4!} \frac{d^4 g}{dx^4}(x) + \varepsilon(x)
\end{align*}
$$

(5.7)

where $\varepsilon(x)$ is a residual error term and $\kappa_1(G)$ has been set equal to $\kappa_1(F)$.  

Proof. See the Appendix in [Jarrow and Rudd, 1982].

Hence it appears that the difference between $f(x)$ and $g(x)$ is expressed as three terms plus an error term. The first term adjusts for difference in variance between the two distributions, while the second and third terms adjust for difference in skewness and kurtosis. The weighting factors are respectively the second, third and fourth derivatives of $g$ with respect to $x$.

The magnitude of the error term is of course very interesting but unfortunately nothing can be said about this in general. It seems plausible though that the more cumulants we include, the better the approximation is. We shall take a closer look at this problem in Section 5.3.
5.2 Algorithm

5.2.1 Model

As before, the stock price is given by

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t$$  (5.8)

and the arithmetic average is

$$A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i}$$  (5.9)

where the averaging dates are

$$t_i = ih, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n$$  (5.10)

The option payoff at maturity $T$ is $\max\{A - K, 0\}$, so the option price at time 0 is

$$C = \exp(-rT) E[(A - K)^+]$$

$$= \exp(-rT) \int_{-\infty}^{\infty} (x - K)^+ f_A(x) dx$$

$$= \exp(-rT) \int_{K}^{\infty} (x - K) f_A(x) dx$$  (5.11)

where $f_A$ is the true, but unknown, density function of $A$. We could now substitute $f(x)$ from (5.7) for $f_A(x)$ in (5.11), but for technical reasons we shall first rewrite the average.

Define the relative stock price from time $t_{i-1}$ to $t_i$

$$R_i = \frac{S_i}{S_{t_{i-1}}}, \quad i = 1, 2, ..., n, \quad t_0 = 0$$  (5.12)

and define

$$L_{n+1} = 1$$

$$L_i = 1 + R_i L_{i+1}, \quad i = 2, 3, ..., n$$  (5.13)

(5.14)

Now, we can write

$$S_{t_i} = S_{t_{i-1}} R_i = S_{t_1} R_2 ... R_i, \quad i = 2, 3, ..., n$$  (5.15)
and the average can be written as

\[ A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i} \]
\[ = \frac{1}{n} (S_{t_1} + S_{t_1}R_2 + S_{t_1}R_2R_3 + \ldots + S_{t_1}R_2\ldots R_n) \]
\[ = \frac{S_{t_1}}{n} (1 + R_2 + R_2R_3 + \ldots + R_2\ldots R_n) \]
\[ = \frac{S_{t_1}}{n} (1 + R_2 (1 + R_3 (1 + \ldots (1 + R_n)))) \]
\[ = \frac{S_{t_1}}{n} L_2 \]
\[ = \frac{S_0}{n} R_1 L_2 \]  

(5.16)

We want to get rid of the deterministic constant, so we define the scaled
average \( Y \)

\[ Y = \frac{n}{S_0} A = R_1 L_2 \]  

(5.17)

and the scaled strike \( k \)

\[ k = \frac{n}{S_0} K \]  

(5.18)

### 5.2.2 Moments and cumulants of the true distribution

In order to apply the Edgeworth series expansion (5.7) to the distribution
of \( Y \), we have to calculate the cumulants of \( Y \) which, in turn, requires us to
calculate the moments of \( Y \). From (5.8), we know that

\[ S_{t_i} = S_{t_{i-1}} \exp((r - q - \frac{1}{2} \sigma^2)h + \sigma(W_{t_i} - W_{t_{i-1}})), \quad i = 1, 2, \ldots, n \]  

(5.19)

so \( R_i \) is lognormally distributed

\[ \log R_i \sim N((r - q - \frac{1}{2} \sigma^2)h, \sigma^2h) \]  

(5.20)

and its moments are

\[ E[R_i^m] = \exp((r - q - \frac{1}{2} \sigma^2)hm + \frac{1}{2} \sigma^2hm^2), \quad m = 1, 2, \ldots \]  

(5.21)
5.2. ALGORITHM

As the Brownian Motion has independent increments, $R_1$ and $L_2$ are independent, and

$$E[Y^m] = E[(R_1 L_2)^m] = E[R_1^m] E[L_2^m], \quad m = 1, 2, \ldots$$  \hspace{1cm} (5.22)

From (5.14) we get

$$E[L_i^m] = E[(1 + R_i L_{i+1})^m], \quad i = 2, 3, \ldots, n, \quad m = 1, 2, \ldots$$

As shown in Appendix B, the first four moments of $L_i$ are

$$E[L_i] = E[1 + R_i L_{i+1}]$$ \hspace{1cm} (5.23)
$$E[L_i^2] = 1 + E[R_i^2] E[L_{i+1}^2] + 2 E[R_i] E[L_{i+1}]$$ \hspace{1cm} (5.24)
$$E[L_i^3] = 1 + E[R_i^3] E[L_{i+1}^2] + 3 E[R_i^2] E[L_{i+1}^3] + 3 E[R_i] E[L_{i+1}^2]$$ \hspace{1cm} (5.25)
$$E[L_i^4] = 1 + E[R_i^4] E[L_{i+1}^4] + 4 E[R_i^3] E[L_{i+1}^3] + 6 E[R_i^2] E[L_{i+1}^2] + 4 E[R_i] E[L_{i+1}]$$ \hspace{1cm} (5.26)

By definition of $L_i$, it is necessary to calculate these moments recursively starting with $i = n$ and ending with $i = 2$.

The cumulants can be calculated from the moments as follows

$$\kappa_1(F) = E[Y]$$ \hspace{1cm} (5.27)
$$\kappa_2(F) = E[Y^2] - E[Y]^2$$ \hspace{1cm} (5.28)
$$\kappa_3(F) = E[Y^3] + 2 E[Y^2] E[Y]$$ \hspace{1cm} (5.29)
$$\kappa_4(F) = E[Y^4] - 6 E[Y]^4 - 4 E[Y^3] E[Y]$$ \hspace{1cm} (5.30)


The details can be found in Appendix B.

5.2.3 Approximating distribution

[Turnbull and Wakeman, 1991] have chosen the lognormal distribution as approximating distribution due to its widespread use in modern finance. By construction, the first moment of the approximating distribution $G$ is set equal to the first moment of the true distribution $F$, cf. Theorem 6. The lognormal distribution is characterized by two parameters, so we need another equation to "calibrate" the distribution. There are several ways to do this, one of which is to set the second moments of the two distributions equal, i.e.

$$\alpha_2(G) = \alpha_2(F)$$ \hspace{1cm} (5.31)
Since \( \alpha_1(G) = \alpha_1(F) \), we can also express the above in terms of the cumulants

\[
\kappa_2(G) = \kappa_2(F)
\]  

(5.32)

Intuitively, this should result in a good fit to the true distribution. We expect that matching the first two moments and adjusting for some of the higher moments will explain a substantial part of the influence which the distribution of the average has on the option price.

We have already seen how to calculate the moments of the true distribution, and the moments of the lognormal distribution with parameters \( \nu \) and \( \lambda^2 \) are

\[
\alpha_m(G) = \int_{-\infty}^{\infty} x^m g(x) dx = \exp(\nu m + \frac{1}{2} \lambda^2 m^2), \quad m = 1, 2, \ldots
\]  

(5.33)

To find the parameters of the approximating distribution, we match the first two moments and solve the following system of equations

\[
\begin{align*}
\alpha_1(F) &= \exp(\nu + \frac{1}{2} \lambda^2) \\
\alpha_2(F) &= \exp(2\nu + 2\lambda^2)
\end{align*}
\]  

(5.34)

(5.35)

As shown in Appendix B, the solution is

\[
\begin{align*}
\nu &= 2 \ln \alpha_1(F) - \frac{1}{2} \ln \alpha_2(F) \\
\lambda^2 &= \ln \alpha_2(F) - 2 \ln \alpha_1(F)
\end{align*}
\]  

(5.36)

(5.37)

Once we have calculated the moments of the true distribution, we can calculate the parameters above, which fully determine the approximating distribution. The next step is to calculate the moments \( \alpha_m(G) \) from (5.33) and then the cumulants \( \kappa_m(G) \), which are obtained in a manner similar to (5.27)-(5.30).

The density function \( g(x) \) of the lognormal distribution with parameters \( \nu \) and \( \lambda^2 \) is

\[
g(x) = \frac{1}{\lambda \sqrt{2\pi} x} \exp\left(-\frac{(\nu - \ln x)^2}{2\lambda^2}\right), \quad x > 0
\]

(5.38)

From (5.7) it seems as if we need to find the first four derivatives of this function, but it turns out that we only need two of them. The first derivative of \( g \) with respect to \( x \) is

\[
\frac{dg}{dx}(x) = g(x) \frac{1}{x} \left(\frac{\nu - \ln x}{\lambda^2} - 1\right), \quad x > 0
\]

(5.39)
5.2. ALGORITHM

and the second derivative is

\[
\frac{d^2 g}{dx^2}(x) = g(x) \frac{1}{x^2} \left( \frac{\nu - \ln x}{\lambda^2} - 1 \right) \left( \frac{\nu - \ln x}{\lambda^2} - 2 \right) - \frac{1}{\lambda^2}, \quad x > 0
\]

(5.40)

Once again the details appear from Appendix B.

5.2.4 Option price

We can rewrite (5.11), so the option price is

\[
C = \exp(-rT)E[(A - K)^+]
\]

\[
= \exp(-rT) \frac{S_0}{n} E[(Y - k)^+]
\]

\[
= \exp(-rT) \frac{S_0}{n} \int_k^\infty (x - k)f(x)dx
\]

(5.41)

where \( f \) is the true density function of \( Y \).

**Lemma 7** Applying the Edgeworth series expansion (5.7) to the integral in (5.41) leads to

\[
\int_k^\infty (x - k)f(x)dx = \int_k^\infty (x - k)g(x)dx + \frac{\kappa_2(F) - \kappa_2(G)}{2!}g(k) - \frac{\kappa_3(F) - \kappa_3(G)}{3!} g(k) \frac{dg}{dx}
\]

\[
+ \frac{\kappa_4(F) - \kappa_4(G) + 3(\kappa_2(F) - \kappa_2(G))^2}{4!} \frac{d^2 g}{dx^2}(k)
\]

\[
+ \varepsilon(k)
\]

(5.42)

where \( G \) is the approximating lognormal distribution.

**Proof.** Substituting \( f(x) \) from (5.7) for \( f(x) \) in

\[
\int_k^\infty (x - k)f(x)dx
\]

gives integrals of the type

\[
\int_k^\infty (x - k) \frac{d^m g}{dx^m}(x)dx, \quad m = 2, 3, 4
\]
Using integration by parts, we get

\[
\int_k^\infty (x-k) \frac{d^mg}{dx^m}(x)dx = \lim_{x \to \infty} \left\{ (x-k) \frac{d^{m-1}g}{dx^{m-1}}(x) \right\} - k \lim_{x \to \infty} \frac{d^{m-1}g}{dx^{m-1}}(k) - \int_k^\infty \frac{d^{m-1}g}{dx^{m-1}}(x)dx
\]

\[
= \lim_{x \to \infty} \left\{ x \frac{d^{m-1}g}{dx^{m-1}}(x) \right\} - (k \lim_{x \to \infty} \frac{d^{m-2}g}{dx^{m-2}}(x) - \frac{d^{m-2}g}{dx^{m-2}}(k))
\]

\[
= \frac{d^{m-2}g}{dx^{m-2}}(k)
\]

where all the limits are zero, because the lognormal density \(g(x)\) tends to zero in the following sense

\[
\lim_{x \to \infty} x^u g(x) = 0, \quad \forall u > 0
\]

which also makes the derivatives tend to zero. Now, (5.42) follows easily. ■

The next theorem tells us how to calculate the approximate option price.

**Theorem 8** The Turnbull and Wakeman price at time 0 of the Asian option with maturity \(T\) and strike \(K\) is

\[
C = \exp(-rT) \frac{S_0}{n} \left\{ \exp\left(\nu + \frac{1}{2} \lambda^2\right) \Phi(d_1) - k \Phi(d_2) + \frac{\kappa_2(G) - \kappa_2(F)}{2!} g(k) - \frac{\kappa_3(G) - \kappa_3(F)}{3!} \frac{dg}{dx}(k) 
\]

\[
+ \frac{\kappa_4(F) - \kappa_4(G) + 3(\kappa_2(F) - \kappa_2(G))^2}{4!} \frac{d^2g}{dx^2}(k) \right\}
\]

(5.43)

where \(\Phi\) is the standard normal distribution function and

\[
d_1 = \frac{\nu + \lambda^2 - \ln k}{\lambda} \quad (5.44)
\]

\[
d_2 = d_1 - \lambda \quad (5.45)
\]

Note that the term with the second cumulants in (5.43) equals zero in our setup (by choice of approximating distribution) and can therefore be removed from the expression.
5.2. ALGORITHM

Proof. As shown below

\[ \int_k^\infty (x - k)g(x)dx = \exp(\nu + \frac{1}{2}\lambda^2)\Phi\left(\frac{\nu + \lambda^2 - \ln k}{\lambda}\right) - k\Phi\left(\frac{\nu - \ln k}{\lambda}\right) \]

Combining this with (5.41) and (5.42) gives the result in (5.43).
First, we split the integral into two integrals

\[ \int_k^\infty (x - k)g(x)dx = \int_k^\infty xg(x)dx - k \int_k^\infty g(x)dx \]

Using the change of variables

\[ y = \frac{\nu + \lambda^2 - \ln x}{\lambda}, \quad \frac{dy}{dx} = -\frac{1}{\lambda x} \]

and the density (5.38), the first integral is

\[ \int_k^\infty xg(x)dx = \int_k^\infty x \frac{1}{\lambda\sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{(\nu - \ln x)^2}{2\lambda^2}\right)dx \]
\[ = \int_k^\infty x \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{1}{2}\left(\frac{\nu - \ln x}{\lambda}\right)^2\right)dx \]
\[ = \int_{\nu + \lambda^2 - \ln k}^{\nu + \lambda^2 - \ln k} - \exp(\nu + \lambda^2 - \lambda y) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y - \lambda)^2\right)dy \]
\[ = \int_{\nu + \lambda^2 - \ln k}^{\nu + \lambda^2 - \ln k} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2 - \frac{1}{2}\lambda^2 + \lambda y + \nu + \lambda^2 - \lambda y\right)dy \]
\[ = \exp(\nu + \frac{1}{2}\lambda^2) \int_{\nu + \lambda^2 - \ln k}^{\nu + \lambda^2 - \ln k} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)dy \]
\[ = \exp(\nu + \frac{1}{2}\lambda^2)\Phi\left(\frac{\nu + \lambda^2 - \ln k}{\lambda}\right) \]

Likewise, we make the change of variables

\[ z = \frac{\nu - \ln x}{\lambda}, \quad \frac{dz}{dx} = -\frac{1}{\lambda x} \]
and the second integral is then

\[
\int_k^\infty g(x)dx = \int_k^\infty \frac{1}{\lambda \sqrt{2\pi} x} \exp\left(\frac{-(\nu - \ln x)^2}{2\lambda^2}\right)dx
\]

\[
= \int_k^\infty \frac{1}{\sqrt{2\pi} \lambda x} \exp\left(-\frac{1}{2} \left(\frac{\nu - \ln x}{\lambda}\right)^2\right)dx
\]

\[
= \int_{\frac{\nu - \ln k}{\lambda}}^{-\infty} -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right)dz
\]

\[
= \int_{-\infty}^{\frac{\nu - \ln k}{\lambda}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right)dz
\]

\[
= \Phi\left(\frac{\nu - \ln k}{\lambda}\right)
\]

which completes the proof. ■

5.2.5 Method overview

The Turnbull and Wakeman algorithm for pricing Asian options can be summarized as follows

- Calculate the first four moments of the true distribution of the (scaled) average stock price.

- Match the first two moments of the approximating lognormal distribution to the true distribution.

- Calculate the third and fourth moments of the approximating distribution.

- Find the first and second derivatives of the approximating density function.

- Calculate the first four cumulants of both distributions.

- Calculate the Asian option price using the generalized Edgeworth series expansion.

5.3 Numerical results

Table 5.1 contains option prices for different values of stock price volatility \(\sigma\), strike price \(K\), number of prices in the average \(n\) and maturity \(T\). The Monte
5.3. **NUMERICAL RESULTS**

Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

\[ q = 0, \quad r = 0.05, \quad S_0 = 100 \quad (5.46) \]

The Turnbull and Wakeman price is within 1 cent of the Monte Carlo estimate for all values of \( K \) and \( n \) and for small values of \( \sigma \) and \( T \). The accuracy deteriorates for volatilities larger than 30% p.a. and especially for maturities longer than \( 1\frac{1}{2} \) years.

We stated in Section 5.1 that nothing can be said about the error term \( \varepsilon(x) \) in general. In the special case where all the moments exist (as in our setup), we have

\[ \varepsilon_N(x) \to 0 \text{ uniformly as } N \to \infty \quad (5.47) \]

where \( N \) represents the number of terms included in the series expansion (see [Jarrow and Rudd, 1982]). This is very reassuring but for all practical purposes \( N \) is finite, so how does it influence the accuracy if we add or remove correction terms? To investigate this we have included the components of the pricing expression (5.43) in Table 5.2. The notation is

\[
\begin{align*}
term2 & = -\exp(-rT) \frac{S_0}{n} \frac{\kappa_3(F) - \kappa_3(G)}{3!} \frac{dg}{dx}(k), \\
term3 & = \exp(-rT) \frac{S_0}{n} \frac{\kappa_4(F) - \kappa_4(G) + 3(\kappa_2(F) - \kappa_2(G))^2}{4!} \frac{d^2g}{dx^2}(k), \\
dg & = \frac{dg}{dx}(k), \quad d^2g = \frac{d^2g}{dx^2}(k), \quad c3/6 = \frac{\kappa_3(F) - \kappa_3(G)}{3!}, \\
c4/24 & = \frac{\kappa_4(F) - \kappa_4(G) + 3(\kappa_2(F) - \kappa_2(G))^2}{4!} \\
\end{align*}
\]

(5.48)

Note that the correction term \( term2 \) is small (less than 3 cents) except for large volatilities whereas \( term3 \) plays an important role for large values of both \( \sigma \) and \( T \). This is a consequence of a considerable difference in kurtosis between the true distribution and the approximating lognormal distribution as seen in the column labelled \( c4/24 \).

The last two columns of Table 5.1 show the Turnbull and Wakeman price with no correction for differences in fourth cumulants (TW3c) and with no corrections at all (Levy). The last column is labelled Levy because this pricing method is equivalent to the one used in [Levy, 1992]. He also refers to it as the ”Wilkinson approximation” (setting the first two moments equal).
Figure 5.1 and Figure 5.2 display these prices less the Monte Carlo estimate for different values of $\sigma$, $T$ and $K$.

Figure 5.1 shows that the Turnbull and Wakeman algorithm performs better than the other two and that not adjusting for cumulant differences (Levy) is better than only adjusting for third cumulants (TW3c). It also appears from Figure 5.1 that the TW algorithm is very sensitive to increases in maturity and should definitely not be used for maturities longer than 3 years. For $T > 3.5$ both Levy and TW3c perform better (but not well!) which is due to the large downward kurtosis correction $\textit{term}3$, cf. Table 5.2.

Figure 5.2 is very interesting. We see that the TW price is within 1 cent of the MC estimate for all strike prices whereas TW3c varies up to 2 cents and Levy varies up to 3 cents. It depends on $K$ which method is the better one, and hence we should not use (5.47) in combination with intuition to say that adding more correction terms gives better accuracy. When the option is deep-in-the-money TW overprices it and then underprices it as $K$ is getting closer to $S_0$. This changes again when the option is out-of-the-money. To some extent, TW3c behaves in the opposite way while Levy is too large for in-the-money and at-the-money options and too small for out-of-the-money options. The explanations of these fluctuations can be found in Table 5.2 in the columns labelled $dg$ and $d^2g$. The variations of the first and second order derivatives of the approximating density function cause the correction terms $\textit{term}2$ and $\textit{term}3$ to change sign and magnitude. Figure 5.2 shows that $\textit{term}2$ is negative when $K < S_0$ and positive when $K > S_0$. $\textit{term}3$ is negative when $K$ is near $S_0$ and positive elsewhere.

Finally, we note from Table 5.1 that increasing the number of prices in the average $n$ does not affect the accuracy of the Turnbull and Wakeman algorithm, but the option prices decrease towards the continuous-time-averaging prices.
5.3. **NUMERICAL RESULTS**

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$K$</th>
<th>$n$</th>
<th>$T$</th>
<th>MC</th>
<th>sd</th>
<th>TW</th>
<th>TW3c</th>
<th>Levy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>2.93</td>
<td>0.001</td>
<td>2.93</td>
<td>2.93</td>
<td>2.93</td>
</tr>
<tr>
<td>0.10</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>3.90</td>
<td>0.0004</td>
<td>3.90</td>
<td>3.91</td>
<td>3.91</td>
</tr>
<tr>
<td>0.15</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>5.01</td>
<td>0.0009</td>
<td>5.01</td>
<td>5.02</td>
<td>5.02</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.15</td>
<td>6.17</td>
<td>6.17</td>
</tr>
<tr>
<td>0.25</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>7.31</td>
<td>0.0024</td>
<td>7.30</td>
<td>7.35</td>
<td>7.34</td>
</tr>
<tr>
<td>0.30</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>8.47</td>
<td>0.0035</td>
<td>8.46</td>
<td>8.53</td>
<td>8.52</td>
</tr>
<tr>
<td>0.35</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>9.64</td>
<td>0.0049</td>
<td>9.61</td>
<td>9.74</td>
<td>9.70</td>
</tr>
<tr>
<td>0.40</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>10.80</td>
<td>0.0065</td>
<td>10.77</td>
<td>10.95</td>
<td>10.89</td>
</tr>
<tr>
<td>0.45</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>11.96</td>
<td>0.0085</td>
<td>11.92</td>
<td>12.18</td>
<td>12.08</td>
</tr>
<tr>
<td>0.50</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>13.11</td>
<td>0.0107</td>
<td>13.07</td>
<td>13.44</td>
<td>13.28</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.5</td>
<td>4.11</td>
<td>0.0007</td>
<td>4.11</td>
<td>4.12</td>
<td>4.12</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.0</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.15</td>
<td>6.17</td>
<td>6.17</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.5</td>
<td>7.84</td>
<td>0.0025</td>
<td>7.83</td>
<td>7.87</td>
<td>7.87</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.0</td>
<td>9.32</td>
<td>0.0034</td>
<td>9.30</td>
<td>9.36</td>
<td>9.37</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.5</td>
<td>10.66</td>
<td>0.0045</td>
<td>10.62</td>
<td>10.72</td>
<td>10.73</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.0</td>
<td>11.90</td>
<td>0.0055</td>
<td>11.83</td>
<td>11.98</td>
<td>11.99</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.5</td>
<td>13.05</td>
<td>0.0066</td>
<td>12.95</td>
<td>13.15</td>
<td>13.16</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.0</td>
<td>14.14</td>
<td>0.0078</td>
<td>13.99</td>
<td>14.25</td>
<td>14.26</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.5</td>
<td>15.16</td>
<td>0.0090</td>
<td>14.96</td>
<td>15.29</td>
<td>15.30</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>5.0</td>
<td>16.12</td>
<td>0.0102</td>
<td>15.86</td>
<td>16.27</td>
<td>16.29</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.15</td>
<td>6.17</td>
<td>6.17</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>24</td>
<td>1</td>
<td>5.96</td>
<td>0.0016</td>
<td>5.96</td>
<td>5.98</td>
<td>5.98</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>36</td>
<td>1</td>
<td>5.90</td>
<td>0.0016</td>
<td>5.89</td>
<td>5.91</td>
<td>5.91</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>48</td>
<td>1</td>
<td>5.86</td>
<td>0.0016</td>
<td>5.86</td>
<td>5.88</td>
<td>5.88</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>60</td>
<td>1</td>
<td>5.84</td>
<td>0.0016</td>
<td>5.84</td>
<td>5.86</td>
<td>5.86</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>70</td>
<td>1</td>
<td>31.16</td>
<td>0.0015</td>
<td>31.16</td>
<td>31.16</td>
<td>31.16</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>75</td>
<td>1</td>
<td>26.42</td>
<td>0.0015</td>
<td>26.42</td>
<td>26.41</td>
<td>26.42</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>80</td>
<td>1</td>
<td>21.72</td>
<td>0.0015</td>
<td>21.72</td>
<td>21.71</td>
<td>21.73</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>85</td>
<td>1</td>
<td>17.16</td>
<td>0.0015</td>
<td>17.16</td>
<td>17.15</td>
<td>17.19</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>90</td>
<td>1</td>
<td>12.92</td>
<td>0.0015</td>
<td>12.91</td>
<td>12.92</td>
<td>12.95</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>95</td>
<td>1</td>
<td>9.19</td>
<td>0.0015</td>
<td>9.18</td>
<td>9.20</td>
<td>9.22</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.15</td>
<td>6.17</td>
<td>6.17</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>105</td>
<td>1</td>
<td>3.87</td>
<td>0.0016</td>
<td>3.87</td>
<td>3.89</td>
<td>3.87</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>110</td>
<td>1</td>
<td>2.29</td>
<td>0.0015</td>
<td>2.30</td>
<td>2.30</td>
<td>2.28</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>115</td>
<td>1</td>
<td>1.28</td>
<td>0.0015</td>
<td>1.28</td>
<td>1.28</td>
<td>1.25</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>120</td>
<td>1</td>
<td>0.67</td>
<td>0.0013</td>
<td>0.68</td>
<td>0.67</td>
<td>0.65</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>125</td>
<td>1</td>
<td>0.34</td>
<td>0.0012</td>
<td>0.34</td>
<td>0.33</td>
<td>0.32</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>130</td>
<td>1</td>
<td>0.16</td>
<td>0.0010</td>
<td>0.16</td>
<td>0.16</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 5.1: Asian option prices for different values of volatility, strike price, number of averaging times and maturity. MC: Monte Carlo simulation. sd: Standard deviation of estimator. TW: Turnbull and Wakeman price. (Continued in Table 5.2).
<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>K</th>
<th>n</th>
<th>T</th>
<th>term2</th>
<th>term3</th>
<th>c3/6</th>
<th>dg</th>
<th>c4/24</th>
<th>$d^2g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>0.00</td>
<td>0.0001</td>
<td>1.6724</td>
<td>0.0000</td>
<td>-1.7742</td>
</tr>
<tr>
<td>0.10</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>0.00</td>
<td>0.0021</td>
<td>0.2330</td>
<td>0.0003</td>
<td>-0.8139</td>
</tr>
<tr>
<td>0.15</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>-0.01</td>
<td>0.0109</td>
<td>0.0485</td>
<td>0.0037</td>
<td>-0.2795</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.0353</td>
<td>0.0062</td>
<td>0.0216</td>
<td>-0.1214</td>
</tr>
<tr>
<td>0.25</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>-0.04</td>
<td>0.0888</td>
<td>-0.0064</td>
<td>0.0865</td>
<td>-0.0615</td>
</tr>
<tr>
<td>0.30</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.02</td>
<td>-0.07</td>
<td>0.1912</td>
<td>-0.0106</td>
<td>0.2739</td>
<td>-0.0345</td>
</tr>
<tr>
<td>0.35</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.03</td>
<td>-0.12</td>
<td>0.3702</td>
<td>-0.0118</td>
<td>0.7403</td>
<td>-0.0208</td>
</tr>
<tr>
<td>0.40</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.06</td>
<td>-0.19</td>
<td>0.6647</td>
<td>-0.0118</td>
<td>1.7883</td>
<td>-0.0131</td>
</tr>
<tr>
<td>0.45</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.10</td>
<td>-0.27</td>
<td>1.1286</td>
<td>-0.0114</td>
<td>3.9766</td>
<td>-0.0085</td>
</tr>
<tr>
<td>0.50</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.16</td>
<td>-0.37</td>
<td>1.8367</td>
<td>-0.0108</td>
<td>8.3069</td>
<td>-0.0056</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.5</td>
<td>0.00</td>
<td>-0.01</td>
<td>0.0082</td>
<td>0.0092</td>
<td>0.0024</td>
<td>-0.3485</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.0</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.0353</td>
<td>0.0062</td>
<td>0.0216</td>
<td>-0.1214</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.5</td>
<td>0.00</td>
<td>-0.04</td>
<td>0.0856</td>
<td>0.0049</td>
<td>0.0813</td>
<td>-0.0651</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.0</td>
<td>0.01</td>
<td>-0.07</td>
<td>0.1639</td>
<td>0.0041</td>
<td>0.2150</td>
<td>-0.0416</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.5</td>
<td>0.01</td>
<td>-0.10</td>
<td>0.2761</td>
<td>0.0035</td>
<td>0.4685</td>
<td>-0.0293</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.0</td>
<td>0.01</td>
<td>-0.14</td>
<td>0.4287</td>
<td>0.0030</td>
<td>0.9038</td>
<td>-0.0220</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.5</td>
<td>0.01</td>
<td>-0.19</td>
<td>0.6295</td>
<td>0.0027</td>
<td>1.6032</td>
<td>-0.0172</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.0</td>
<td>0.01</td>
<td>-0.25</td>
<td>0.8871</td>
<td>0.0024</td>
<td>2.6749</td>
<td>-0.0139</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.5</td>
<td>0.02</td>
<td>-0.32</td>
<td>1.2118</td>
<td>0.0022</td>
<td>4.2597</td>
<td>-0.0115</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>5.0</td>
<td>0.02</td>
<td>-0.41</td>
<td>1.6152</td>
<td>0.0020</td>
<td>6.5391</td>
<td>-0.0096</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.0353</td>
<td>0.0062</td>
<td>0.0216</td>
<td>-0.1214</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>24</td>
<td>1</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.2826</td>
<td>0.0018</td>
<td>0.3265</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>36</td>
<td>1</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.9529</td>
<td>0.0008</td>
<td>1.6194</td>
<td>-0.0051</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>48</td>
<td>1</td>
<td>0.00</td>
<td>-0.02</td>
<td>2.2576</td>
<td>0.0005</td>
<td>5.0648</td>
<td>-0.0022</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>60</td>
<td>1</td>
<td>0.00</td>
<td>-0.02</td>
<td>4.4076</td>
<td>0.0003</td>
<td>12.2871</td>
<td>-0.0011</td>
</tr>
<tr>
<td>0.20</td>
<td>70</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>0.00</td>
<td>0.0353</td>
<td>0.0105</td>
<td>0.0216</td>
<td>0.0248</td>
</tr>
<tr>
<td>0.20</td>
<td>75</td>
<td>12</td>
<td>1</td>
<td>-0.01</td>
<td>0.01</td>
<td>0.0353</td>
<td>0.0346</td>
<td>0.0216</td>
<td>0.0566</td>
</tr>
<tr>
<td>0.20</td>
<td>80</td>
<td>12</td>
<td>1</td>
<td>-0.02</td>
<td>0.01</td>
<td>0.0353</td>
<td>0.0758</td>
<td>0.0216</td>
<td>0.0751</td>
</tr>
<tr>
<td>0.20</td>
<td>85</td>
<td>12</td>
<td>1</td>
<td>-0.03</td>
<td>0.01</td>
<td>0.0353</td>
<td>0.1145</td>
<td>0.0216</td>
<td>0.0442</td>
</tr>
<tr>
<td>0.20</td>
<td>90</td>
<td>12</td>
<td>1</td>
<td>-0.03</td>
<td>0.01</td>
<td>0.0353</td>
<td>0.1193</td>
<td>0.0216</td>
<td>-0.0324</td>
</tr>
<tr>
<td>0.20</td>
<td>95</td>
<td>12</td>
<td>1</td>
<td>-0.02</td>
<td>0.02</td>
<td>0.0353</td>
<td>0.0768</td>
<td>0.0216</td>
<td>-0.1034</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.0353</td>
<td>0.0062</td>
<td>0.0216</td>
<td>-0.1214</td>
</tr>
<tr>
<td>0.20</td>
<td>105</td>
<td>12</td>
<td>1</td>
<td>0.02</td>
<td>-0.01</td>
<td>0.0353</td>
<td>-0.0577</td>
<td>0.0216</td>
<td>-0.0850</td>
</tr>
<tr>
<td>0.20</td>
<td>110</td>
<td>12</td>
<td>1</td>
<td>0.03</td>
<td>0.00</td>
<td>0.0353</td>
<td>-0.0916</td>
<td>0.0216</td>
<td>-0.0279</td>
</tr>
<tr>
<td>0.20</td>
<td>115</td>
<td>12</td>
<td>1</td>
<td>0.03</td>
<td>0.00</td>
<td>0.0353</td>
<td>-0.0938</td>
<td>0.0216</td>
<td>0.0170</td>
</tr>
<tr>
<td>0.20</td>
<td>120</td>
<td>12</td>
<td>1</td>
<td>0.02</td>
<td>0.01</td>
<td>0.0353</td>
<td>-0.0763</td>
<td>0.0216</td>
<td>0.0373</td>
</tr>
<tr>
<td>0.20</td>
<td>125</td>
<td>12</td>
<td>1</td>
<td>0.01</td>
<td>0.01</td>
<td>0.0353</td>
<td>-0.0529</td>
<td>0.0216</td>
<td>0.0380</td>
</tr>
<tr>
<td>0.20</td>
<td>130</td>
<td>12</td>
<td>1</td>
<td>0.01</td>
<td>0.01</td>
<td>0.0353</td>
<td>-0.0325</td>
<td>0.0216</td>
<td>0.0293</td>
</tr>
</tbody>
</table>

Table 5.2: (Continued from Table 5.1). Components of the Turnbull and Wakeman pricing expression (5.43). See (5.48) for an explanation of the column headers.
Figure 5.1: Option prices subtracted by Monte Carlo estimate. TW: Turnbull and Wakeman price. TW3c: TW with correction for differences in third cumulants only. Levy: TW without correction for differences in cumulants.
Figure 5.2: Top: Option prices subtracted by Monte Carlo estimate. Bottom: Contribution of correction terms to Turnbull and Wakeman price. See (5.48).
5.3. NUMERICAL RESULTS

5.3.1 Conclusion

We have seen that the accuracy of the Turnbull and Wakeman algorithm is good when the stock price volatility is low and when the maturity of the option is short. The algorithm should be used with caution when the volatility exceeds 20-30% p.a. and it should never be used to price Asian options with a maturity longer than 2-3 years. The generalized Edgeworth series expansion adjusts for differences in kurtosis (and skewness), which results in huge pricing errors for options with long maturities. A possible way to fix this is to set the fourth moments equal instead of the second moments (5.31), but that is outside the scope of this thesis. Finally, we conclude that adding more terms to the Edgeworth series expansion does not necessarily improve the accuracy of the approximation.
Chapter 6

Milevsky and Posner method

[Milevsky and Posner, 1998] are inspired by previous attempts to find a good approximation for the distribution of the arithmetic average and use this to price Asian options. Some of the most frequently quoted articles in this context are [Levy, 1992], [Vorst, 1992] and [Turnbull and Wakeman, 1991]. Milevsky and Posner argue that the reciprocal gamma distribution should be preferred to the lognormal distribution as approximating distribution. This is due to the fact that the infinite sum of correlated lognormal random variables is reciprocally gamma distributed. The resulting pricing formula bears a close resemblance to the Black-Scholes formula and the method is very fast and easy to implement.

6.1 Reciprocal gamma distribution

6.1.1 Gamma distribution

Let $X \sim \Gamma(\alpha, \beta)$ denote that the random variable $X$ is gamma distributed with parameters $\alpha > 0$ and $\beta > 0$. The density function $g$ of $X$ is

$$g(x) = \Gamma(\alpha)^{-1} \beta^{-\alpha} x^{\alpha-1} \exp(-\beta^{-1} x), \quad x > 0$$

(6.1)

where $\Gamma$ is the gamma function (see [Hoffmann-Jørgensen, 1994a] p. 216). To emphasize what the parameters are, we shall also use the notation $g(x|\alpha, \beta)$ and likewise for the distribution function $G(x|\alpha, \beta)$. Some useful properties are collected in the following proposition.
Proposition 9 Let $\alpha > 0$ and $\beta > 0$. Then
\[
G(x|\alpha, \beta) = G\left(\frac{x}{\beta}\right|\alpha, 1), \quad \forall x > 0 \tag{6.2}
\]
\[
g(x|\alpha, \beta) = \frac{1}{\beta}g\left(\frac{x}{\beta}\right|\alpha, 1), \quad \forall x > 0 \tag{6.3}
\]
\[
g(x|\alpha, \beta) = \frac{x}{\beta(\alpha - 1)}g(x|\alpha - 1, \beta), \quad \forall x > 0, \quad \forall \alpha > 1 \tag{6.4}
\]
Proof. See Appendix C. \hfill \blacksquare

6.1.2 Reciprocal gamma distribution

Define $Y = \frac{1}{X}$. The random variable $Y$ is reciprocally gamma distributed with parameters $\alpha$ and $\beta$. The distribution function $G_R$ of $Y$ is related to the distribution function $G$ of $X$ in the following way
\[
G_R(y) = P(Y \leq y) = P\left(\frac{1}{Y} \geq \frac{1}{y}\right) = P\left(X \geq \frac{1}{y}\right) = 1 - P\left(X \leq \frac{1}{y}\right)
\]
\[
= 1 - G\left(\frac{1}{y}\right), \quad y > 0 \tag{6.5}
\]
Since the gamma distribution is absolutely continuous, $\frac{d}{dy}(G(y)) = g(y)$, and therefore the density function $g_R$ of the reciprocal gamma distribution is
\[
g_R(y) = \frac{d}{dy}(G_R(y)) = \frac{d}{dy}(1 - G\left(\frac{1}{y}\right)) = 0 - \frac{-1}{y^2}g\left(\frac{1}{y}\right)
\]
\[
= \frac{1}{y^2}g\left(\frac{1}{y}\right), \quad y > 0 \tag{6.6}
\]
It follows that all results with the reciprocal gamma distribution can be rewritten and expressed by means of the gamma distribution instead. This is convenient since the gamma distribution is included in many software packages like MS Excel etc.

As shown in Appendix C, the moments $M_i$ of the reciprocal gamma distribution are
\[
M_i = E[Y^i] = E\left[\frac{1}{X^i}\right] = \frac{1}{\beta^i(\alpha - 1)(\alpha - 2)\ldots(\alpha - i)}, \quad i \in \mathbb{N}, \quad i < \alpha \tag{6.7}
\]
and the parameters $\alpha$ and $\beta$ can be calculated from the first two moments which, therefore, fully specify the distribution
\[
\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2} \tag{6.8}
\]
\[
\beta = \frac{M_2 - M_1^2}{M_1 M_2} \tag{6.9}
\]
6.2 Method

6.2.1 Model

The stock price is given by

\[ dS_t = (r - q)S_t dt + \sigma S_t dW_t \]  

(6.10)

and the arithmetic average is

\[ A = \frac{1}{n} \sum_{i=1}^{n} S_t \]  

(6.11)

where the averaging dates are

\[ t_i = i h, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n \]  

(6.12)

For continuous averaging, we have

\[ A = \frac{1}{T} \int_{0}^{T} S_u du \]  

(6.13)

The option payoff at maturity \( T \) is \( \max\{A - K, 0\} \), so the option price at time 0 is

\[
C = \exp(-rT)E[(A - K)^+] \\
= \exp(-rT) \int_{-\infty}^{\infty} (y - K)^+ f(y) dy \\
= \exp(-rT) \int_{K}^{\infty} (y - K) f(y) dy
\]  

(6.14)

where \( f \) is the true, but unknown, density function of \( A \). We approximate \( f \) with the density function \( g_R \) of a reciprocal gamma distribution. Fortunately we can calculate the moments of the true distribution, so we can match these with the moments of the approximating distribution in order to get a good fit. The approximate price is then

\[ C = \exp(-rT) \int_{K}^{\infty} (y - K) g_R(y) dy \]  

(6.15)
6.2.2 Discrete averaging

Define

$$I = \frac{1}{S_0} \sum_{i=1}^{n} S_{t_i}$$

(6.16)

which is the sum of the averaging date prices scaled by the current price. Later in this chapter, we shall see that the limiting distribution (as $T \to \infty$) of the continuous averaging version of $I$ is a reciprocal gamma distribution. This is the theoretical justification for approximating the average with the reciprocal gamma distribution. The relationship between $I$ and the arithmetic average $A$ is

$$A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i} = \frac{S_0}{n} I$$

(6.17)

The first two risk neutral moments of the average are for $r \neq q$

$$E[A] = \frac{S_0}{n} e^{(r-q)h} \left( \frac{1 - e^{(r-q)n^h}}{1 - e^{(r-q)h}} \right)$$

(6.18)

$$E[A^2] = \frac{S_0^2}{n^2} (a_1 - a_2 + a_3 - a_4)$$

(6.19)

where

$$a_1 = \frac{e^{(2(r-q)+\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h})}$$

(6.20)

$$a_2 = \frac{e^{(r-q)(n+2)+\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h})}$$

(6.21)

$$a_3 = \frac{e^{(3(r-q)+\sigma^2)h} - e^{(r-q)(n+2)+\sigma^2)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h})}$$

(6.22)

$$a_4 = \frac{e^{(4(r-q)+2\sigma^2)h} - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q+\sigma^2)h})(1 - e^{(r-q+\sigma^2)h})}$$

(6.23)

(6.18)-(6.19) are modified versions of the formulae in [Levy, 1992] (see Appendix C for details). We choose the parameters of the approximating reciprocal gamma distribution so its moments are equal to the above moments of the true distribution. From (6.8) and (6.9), we get

$$\alpha = \frac{2E[A^2] - E[A]^2}{E[A^2] - E[A]^2}$$

(6.24)

$$\beta = \frac{E[A^2] - E[A]^2}{E[A]E[A^2]}$$

(6.25)

We can now calculate the option price from the following theorem.
6.2. METHOD

**Theorem 10**  The Milevsky and Posner price at time 0 of the Asian option with maturity $T$ and strike $K$ is

$$C = \exp(-rT)E[A]G\left(\frac{1}{K}\mid \alpha - 1, \beta\right) - \exp(-rT)KG\left(\frac{1}{K}\mid \alpha, \beta\right) \quad (6.26)$$

where $G(\cdot | \alpha, \beta)$ is the distribution function of the gamma distribution with parameters $\alpha$ and $\beta$ obtained from (6.24) and (6.25).

**Proof.** From (6.15) and (6.6)

$$C = \exp(-rT)\int_K^\infty (y - K)g_R(y|\alpha, \beta)dy$$

$$= \exp(-rT)\int_K^\infty \frac{1}{y^2}g\left(\frac{1}{y}\mid \alpha, \beta\right)dy$$

We change variables $x = \frac{1}{y}$ so $\frac{dx}{dy} = -\frac{1}{y^2}$, which leads to

$$C = \exp(-rT)\int_0^\infty -\left(\frac{1}{x} - K\right)g(x|\alpha, \beta)dx$$

$$= \exp(-rT)\int_0^\infty \left(\frac{1}{x} - K\right)g(x|\alpha, \beta)dx$$

$$= \exp(-rT)\int_0^\infty \frac{1}{x}g(x|\alpha, \beta)dx - \exp(-rT)K\int_0^\infty g(x|\alpha, \beta)dx$$

$$= \exp(-rT)\int_0^\infty \frac{1}{\beta(\alpha - 1)}g(x|\alpha - 1, \beta)dx - \exp(-rT)KG\left(\frac{1}{K}\mid \alpha, \beta\right)$$

$$= \exp(-rT)\frac{1}{\beta(\alpha - 1)}G\left(\frac{1}{K}\mid \alpha - 1, \beta\right) - \exp(-rT)KG\left(\frac{1}{K}\mid \alpha, \beta\right)$$

where we have used (6.4). Since $M_1 = E[A]$ by choice of approximating distribution and $M_1 = \frac{1}{\beta(\alpha - 1)}$ from (6.7), we get the wanted expression

$$C = \exp(-rT)E[A]G\left(\frac{1}{K}\mid \alpha - 1, \beta\right) - \exp(-rT)KG\left(\frac{1}{K}\mid \alpha, \beta\right)$$

Note how the pricing formula (6.26) is similar to the Black-Scholes formula (2.3) with $\Phi(d_1)$ and $\Phi(d_2)$ replaced with $G\left(\frac{1}{K}\mid \alpha - 1, \beta\right)$ and $G\left(\frac{1}{K}\mid \alpha, \beta\right)$.

6.2.3 Method overview

The Milevsky and Posner algorithm can be summarized as follows
CHAPTER 6. MILEVSKY AND POSNER METHOD

- Calculate the first two moments of the arithmetic average of the stock price.
- Match the first two moments of the approximating distribution to the moments above and calculate the parameters of the reciprocal gamma distribution.
- Calculate the Asian option price using Theorem 10.

6.2.4 Continuous averaging

For the sake of completeness, we shall also take a look at the Milevsky and Posner method with continuous averaging. Define the scaled integral

\[ I = \frac{1}{S_0} \int_0^T S_u du \tag{6.27} \]

so the continuous average is

\[ A = \frac{1}{T} \int_0^T S_u du = \frac{S_0}{T} I \tag{6.28} \]

The limiting distribution of \( I \) is the reciprocal gamma distribution as stated in the next theorem.

**Theorem 11** Let

\[ I_\infty = \lim_{T \to \infty} I = \frac{1}{S_0} \lim_{T \to \infty} \int_0^T S_u du \tag{6.29} \]

Assume \( r - q - \frac{1}{2} \sigma^2 < 0 \). Then \( I_\infty \) is reciprocally gamma distributed with parameters \( 1 + \frac{2(q-r)}{\sigma^2} \) and \( \frac{1}{2} \sigma^2 \) which is equivalent to

\[ I_\infty^{-1} \sim \Gamma(1 + \frac{2(q-r)}{\sigma^2}, \frac{1}{2} \sigma^2) \tag{6.30} \]

**Proof.** See Appendix A of [Milevsky and Posner, 1998]

Note that the result is only valid when \( r - q - \frac{1}{2} \sigma^2 < 0 \). This means that we should expect the Milevsky and Posner method to perform best when the volatility \( \sigma \) is high and/or when the stock dividend \( q \) is greater than the interest rate \( r \).
6.3. NUMERICAL RESULTS

According to [Milevsky and Posner, 1998], the first two risk neutral moments of the average are for \( r \neq q \)

\[
E[A] = S_0 \frac{\exp((r - q)T) - 1}{(r - q)T}
\]

\[
E[A^2] = \frac{2S_0^2}{T^2} (\frac{\exp((2(r - q) + \sigma^2)T)}{(r - q + \sigma^2)(2r - 2q + \sigma^2)}
+ \frac{1}{r - q} \frac{1}{2(r - q + \sigma^2)} - \frac{\exp((r - q)T)}{r - q + \sigma^2})
\]

and for \( r = q \)

\[
E[A] = S_0
\]

\[
E[A^2] = \frac{2S_0^2 \exp(\sigma^2 T)}{\sigma^4} (1 - \frac{\exp(\sigma^2 T)}{r - q + \sigma^2})
\]

We can now calculate the parameters \( \alpha \) and \( \beta \) of the approximating reciprocal gamma distribution from (6.24) and (6.25). The pricing formula is given in the following theorem.

**Theorem 12** The Milevsky and Posner price at time 0 of the continuous averaging Asian option with maturity \( T \) and strike \( K \) is

\[
C = S_0 \frac{\exp(-qT) - \exp(-rT)}{(r - q)T} G(\frac{1}{K}|\alpha - 1, \beta)
\]

\[
- \exp(-rT) K G(\frac{1}{K}|\alpha, \beta)
\]

for \( r \neq q \) and

\[
C = S_0 \exp(-rT) G(\frac{1}{K}|\alpha - 1, \beta) - \exp(-rT) K G(\frac{1}{K}|\alpha, \beta)
\]

for \( r = q \).

**Proof.** Insert (6.31) and (6.33) respectively into (6.26). ■

### 6.3 Numerical results

Some of the questions we would like to answer in this section are

1. Is the reciprocal gamma distribution a better choice for approximating the arithmetic average than the lognormal distribution? In other words, is the Milevsky and Posner (MP) method better than the Levy method?
2. How is the accuracy of the MP method affected when we increase the stock dividend $q$?

3. Can we improve the MP method by correcting for differences in skewness and kurtosis as the Turnbull and Wakeman (TW) method does?

4. What is the relationship between the discrete and continuous averaging prices?

Table 6.1 contains option prices for different values of stock price volatility $\sigma$, strike price $K$, number of prices in the average $n$ and maturity $T$. The Monte Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

$$q = 0, \quad r = 0.05, \quad S_0 = 100$$

We have included both discrete and continuous averaging prices, cf. (6.26) and (6.35), as well as prices obtained by using a generalized Edgeworth series expansion (MP2). The reciprocal gamma distribution has all the necessary properties to allow us to develop similar results as in Chapter 5. Thus, from a theoretical viewpoint MP2 has the potential to be just as good as the Turnbull and Wakeman method. Unfortunately this is not the case as we shall see.

To answer the first question we look at Figure 6.1 and Figure 6.2. It appears that the accuracy of the Milevsky and Posner method is more or less similar to the accuracy of the Levy method, cf. Chapter 5. The important difference is that when MP underprices then Levy overprices and vice versa. Based on this, an obvious way to improve the accuracy is to use the average of the two prices. Numerical results (illustrated in Section 7.2 and Section 8.2) show that this approach is better than any of the methods we have seen so far, but also that it can not compete with the Curran method in Chapter 8. Since the MP and Levy methods match the approximating distribution to the first two moments of the true distribution, the price differences observed are caused by the higher moments. Figure 6.2 shows the density of the lognormal distribution subtracted by the density of the reciprocal gamma distribution.

The assumption in Theorem 11 was that $r - q - \frac{1}{2} \sigma^2 < 0$, but Figure 6.1 shows that the MP method is not good for pricing options with high volatilities. Let us then take a look at the stock dividend $q$. Table 6.2 contains prices of Asian options with various strike prices for three different values of the stock dividend. These prices are illustrated in Figure 6.3 where
we see that an increase in $q$ leads to a parallel shift along the first axis. The magnitude of the maximum deviation from the Monte Carlo estimate is approximately the same for different values of $q$, so we conclude that the MP method is not more precise for high stock dividends.

We saw in Chapter 5 that the TW algorithm performed better than the Levy method in most cases. This has inspired us to improve the MP method in a similar fashion. The prices labelled MP2 have been calculated by means of the Turnbull and Wakeman algorithm with a reciprocal gamma distribution as approximating distribution. The moments and cumulants are obtained from (6.24)-(6.25), (6.7) and (5.27)-(5.30) and the derivatives of the reciprocal gamma density appear from Appendix C. Figure 6.1 and Figure 6.2 show that the accuracy has improved for all volatilities and for most of the strike prices. In contrast, it is evident from Figure 6.1 that MP2 is extremely sensitive to increases in maturity and performs worse than the MP method for $T > 2$. Moreover, comparisons (see Section 7.2) show that the TW prices in general are better than the MP2 prices.

Table 6.3 and Figure 6.3 show how the discrete averaging prices converge to the continuous averaging prices for both MP and MP2. Even for 120 prices in the average there is a difference of approximately 4-6 cents, which implies that it is not recommendable to use models with continuous averaging when the option in question is based on a discrete average.
<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>K</th>
<th>n</th>
<th>T</th>
<th>MC</th>
<th>sd</th>
<th>MP</th>
<th>MP c</th>
<th>MP2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>2.93</td>
<td>0.0001</td>
<td>2.93</td>
<td>2.71</td>
<td>n/a</td>
</tr>
<tr>
<td>0.10</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>3.90</td>
<td>0.0004</td>
<td>3.90</td>
<td>3.64</td>
<td>n/a</td>
</tr>
<tr>
<td>0.15</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>5.01</td>
<td>0.0009</td>
<td>5.00</td>
<td>4.68</td>
<td>n/a</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.13</td>
<td>5.75</td>
<td>6.16</td>
</tr>
<tr>
<td>0.25</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>7.31</td>
<td>0.0024</td>
<td>7.27</td>
<td>6.83</td>
<td>7.33</td>
</tr>
<tr>
<td>0.30</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>8.47</td>
<td>0.0035</td>
<td>8.42</td>
<td>7.91</td>
<td>8.50</td>
</tr>
<tr>
<td>0.35</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>9.64</td>
<td>0.0049</td>
<td>9.55</td>
<td>8.98</td>
<td>9.68</td>
</tr>
<tr>
<td>0.40</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>10.80</td>
<td>0.0065</td>
<td>10.68</td>
<td>10.05</td>
<td>10.86</td>
</tr>
<tr>
<td>0.45</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>11.96</td>
<td>0.0085</td>
<td>11.80</td>
<td>11.12</td>
<td>12.04</td>
</tr>
<tr>
<td>0.50</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>13.11</td>
<td>0.0107</td>
<td>12.90</td>
<td>12.17</td>
<td>13.21</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.5</td>
<td>4.11</td>
<td>0.0007</td>
<td>4.10</td>
<td>3.85</td>
<td>n/a</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.0</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.13</td>
<td>5.75</td>
<td>6.16</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.5</td>
<td>7.84</td>
<td>0.0025</td>
<td>7.80</td>
<td>7.30</td>
<td>7.86</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.0</td>
<td>9.32</td>
<td>0.0034</td>
<td>9.26</td>
<td>8.66</td>
<td>9.37</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.5</td>
<td>10.66</td>
<td>0.0045</td>
<td>10.58</td>
<td>9.89</td>
<td>10.75</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.0</td>
<td>11.90</td>
<td>0.0055</td>
<td>11.79</td>
<td>11.02</td>
<td>12.06</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.5</td>
<td>13.05</td>
<td>0.0066</td>
<td>12.92</td>
<td>12.07</td>
<td>13.30</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.0</td>
<td>14.14</td>
<td>0.0078</td>
<td>13.98</td>
<td>13.05</td>
<td>14.51</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.5</td>
<td>15.16</td>
<td>0.0090</td>
<td>14.97</td>
<td>13.97</td>
<td>15.70</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>5.0</td>
<td>16.12</td>
<td>0.0102</td>
<td>15.91</td>
<td>14.84</td>
<td>16.89</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>6.0</td>
<td>16.12</td>
<td>0.0102</td>
<td>15.91</td>
<td>14.84</td>
<td>16.89</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>24</td>
<td>1</td>
<td>5.96</td>
<td>0.0016</td>
<td>5.94</td>
<td>5.75</td>
<td>5.97</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>36</td>
<td>1</td>
<td>5.90</td>
<td>0.0016</td>
<td>5.88</td>
<td>5.75</td>
<td>5.90</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>48</td>
<td>1</td>
<td>5.86</td>
<td>0.0016</td>
<td>5.84</td>
<td>5.75</td>
<td>5.87</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>60</td>
<td>1</td>
<td>5.84</td>
<td>0.0016</td>
<td>5.82</td>
<td>5.75</td>
<td>5.85</td>
</tr>
<tr>
<td>0.20</td>
<td>70</td>
<td>12</td>
<td>1</td>
<td>31.16</td>
<td>0.0015</td>
<td>31.16</td>
<td>30.96</td>
<td>31.16</td>
</tr>
<tr>
<td>0.20</td>
<td>75</td>
<td>12</td>
<td>1</td>
<td>26.42</td>
<td>0.0015</td>
<td>26.41</td>
<td>26.20</td>
<td>26.41</td>
</tr>
<tr>
<td>0.20</td>
<td>80</td>
<td>12</td>
<td>1</td>
<td>21.72</td>
<td>0.0015</td>
<td>21.71</td>
<td>21.49</td>
<td>21.71</td>
</tr>
<tr>
<td>0.20</td>
<td>85</td>
<td>12</td>
<td>1</td>
<td>17.16</td>
<td>0.0015</td>
<td>17.14</td>
<td>16.88</td>
<td>17.17</td>
</tr>
<tr>
<td>0.20</td>
<td>90</td>
<td>12</td>
<td>1</td>
<td>12.92</td>
<td>0.0015</td>
<td>12.88</td>
<td>12.57</td>
<td>12.94</td>
</tr>
<tr>
<td>0.20</td>
<td>95</td>
<td>12</td>
<td>1</td>
<td>9.19</td>
<td>0.0015</td>
<td>9.16</td>
<td>8.79</td>
<td>9.21</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.13</td>
<td>5.75</td>
<td>6.16</td>
</tr>
<tr>
<td>0.20</td>
<td>105</td>
<td>12</td>
<td>1</td>
<td>3.87</td>
<td>0.0016</td>
<td>3.87</td>
<td>3.51</td>
<td>3.87</td>
</tr>
<tr>
<td>0.20</td>
<td>110</td>
<td>12</td>
<td>1</td>
<td>2.29</td>
<td>0.0015</td>
<td>2.31</td>
<td>2.00</td>
<td>2.28</td>
</tr>
<tr>
<td>0.20</td>
<td>115</td>
<td>12</td>
<td>1</td>
<td>1.28</td>
<td>0.0015</td>
<td>1.30</td>
<td>1.07</td>
<td>1.27</td>
</tr>
<tr>
<td>0.20</td>
<td>120</td>
<td>12</td>
<td>1</td>
<td>0.67</td>
<td>0.0013</td>
<td>0.70</td>
<td>0.55</td>
<td>0.67</td>
</tr>
<tr>
<td>0.20</td>
<td>125</td>
<td>12</td>
<td>1</td>
<td>0.34</td>
<td>0.0012</td>
<td>0.36</td>
<td>0.26</td>
<td>0.34</td>
</tr>
<tr>
<td>0.20</td>
<td>130</td>
<td>12</td>
<td>1</td>
<td>0.16</td>
<td>0.0010</td>
<td>0.18</td>
<td>0.12</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 6.1: Option prices for different volatilities, maturities, strike prices and number of prices in the average. MP: Milevsky and Posner. MP c: MP with continuous averaging. MP2: MP with cumulant correction à la Turnbull and Wakeman.
6.3. **NUMERICAL RESULTS**

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$K$</th>
<th>$n$</th>
<th>$q$</th>
<th>$T$</th>
<th>MC</th>
<th>sd</th>
<th>MP</th>
<th>MP c</th>
<th>MP2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>70</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>29.06</td>
<td>0.0013</td>
<td>29.06</td>
<td>29.02</td>
<td>29.05</td>
</tr>
<tr>
<td>0.20</td>
<td>75</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>24.32</td>
<td>0.0013</td>
<td>24.32</td>
<td>24.27</td>
<td>24.31</td>
</tr>
<tr>
<td>0.20</td>
<td>80</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>19.66</td>
<td>0.0013</td>
<td>19.64</td>
<td>19.57</td>
<td>19.65</td>
</tr>
<tr>
<td>0.20</td>
<td>85</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>15.19</td>
<td>0.0013</td>
<td>15.16</td>
<td>15.04</td>
<td>15.20</td>
</tr>
<tr>
<td>0.20</td>
<td>90</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>11.11</td>
<td>0.0013</td>
<td>11.08</td>
<td>10.88</td>
<td>11.13</td>
</tr>
<tr>
<td>0.20</td>
<td>95</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>7.65</td>
<td>0.0013</td>
<td>7.62</td>
<td>7.36</td>
<td>7.66</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>4.93</td>
<td>0.0013</td>
<td>4.92</td>
<td>4.63</td>
<td>4.93</td>
</tr>
<tr>
<td>0.20</td>
<td>105</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>2.97</td>
<td>0.0013</td>
<td>2.98</td>
<td>2.71</td>
<td>2.97</td>
</tr>
<tr>
<td>0.20</td>
<td>110</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>1.68</td>
<td>0.0013</td>
<td>1.71</td>
<td>1.48</td>
<td>1.67</td>
</tr>
<tr>
<td>0.20</td>
<td>115</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>0.90</td>
<td>0.0012</td>
<td>0.92</td>
<td>0.76</td>
<td>0.89</td>
</tr>
<tr>
<td>0.20</td>
<td>120</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>0.45</td>
<td>0.0011</td>
<td>0.48</td>
<td>0.37</td>
<td>0.45</td>
</tr>
<tr>
<td>0.20</td>
<td>125</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>0.22</td>
<td>0.0009</td>
<td>0.24</td>
<td>0.17</td>
<td>0.22</td>
</tr>
<tr>
<td>0.20</td>
<td>130</td>
<td>12</td>
<td>0.04</td>
<td>1</td>
<td>0.10</td>
<td>0.0007</td>
<td>0.11</td>
<td>0.08</td>
<td>0.10</td>
</tr>
<tr>
<td>0.20</td>
<td>70</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>27.01</td>
<td>0.0012</td>
<td>27.01</td>
<td>27.13</td>
<td>27.01</td>
</tr>
<tr>
<td>0.20</td>
<td>75</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>22.29</td>
<td>0.0012</td>
<td>22.28</td>
<td>22.39</td>
<td>22.28</td>
</tr>
<tr>
<td>0.20</td>
<td>80</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>17.67</td>
<td>0.0012</td>
<td>17.65</td>
<td>17.72</td>
<td>17.67</td>
</tr>
<tr>
<td>0.20</td>
<td>85</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>13.31</td>
<td>0.0012</td>
<td>13.28</td>
<td>13.28</td>
<td>13.33</td>
</tr>
<tr>
<td>0.20</td>
<td>90</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>9.44</td>
<td>0.0012</td>
<td>9.41</td>
<td>9.32</td>
<td>9.46</td>
</tr>
<tr>
<td>0.20</td>
<td>95</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>6.27</td>
<td>0.0012</td>
<td>6.24</td>
<td>6.07</td>
<td>6.27</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>3.88</td>
<td>0.0011</td>
<td>3.87</td>
<td>3.66</td>
<td>3.87</td>
</tr>
<tr>
<td>0.20</td>
<td>105</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>2.24</td>
<td>0.0011</td>
<td>2.25</td>
<td>2.05</td>
<td>2.23</td>
</tr>
<tr>
<td>0.20</td>
<td>110</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>1.21</td>
<td>0.0011</td>
<td>1.24</td>
<td>1.07</td>
<td>1.20</td>
</tr>
<tr>
<td>0.20</td>
<td>115</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>0.62</td>
<td>0.0010</td>
<td>0.64</td>
<td>0.53</td>
<td>0.61</td>
</tr>
<tr>
<td>0.20</td>
<td>120</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>0.30</td>
<td>0.0008</td>
<td>0.32</td>
<td>0.24</td>
<td>0.30</td>
</tr>
<tr>
<td>0.20</td>
<td>125</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>0.14</td>
<td>0.0007</td>
<td>0.15</td>
<td>0.11</td>
<td>0.14</td>
</tr>
<tr>
<td>0.20</td>
<td>130</td>
<td>12</td>
<td>0.08</td>
<td>1</td>
<td>0.06</td>
<td>0.0006</td>
<td>0.07</td>
<td>0.05</td>
<td>0.06</td>
</tr>
<tr>
<td>0.20</td>
<td>70</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>25.03</td>
<td>0.0012</td>
<td>25.02</td>
<td>25.29</td>
<td>25.02</td>
</tr>
<tr>
<td>0.20</td>
<td>75</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>20.32</td>
<td>0.0012</td>
<td>20.31</td>
<td>20.56</td>
<td>20.32</td>
</tr>
<tr>
<td>0.20</td>
<td>80</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>15.77</td>
<td>0.0011</td>
<td>15.74</td>
<td>15.94</td>
<td>15.77</td>
</tr>
<tr>
<td>0.20</td>
<td>85</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>11.55</td>
<td>0.0011</td>
<td>11.52</td>
<td>11.62</td>
<td>11.57</td>
</tr>
<tr>
<td>0.20</td>
<td>90</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>7.92</td>
<td>0.0011</td>
<td>7.89</td>
<td>7.88</td>
<td>7.94</td>
</tr>
<tr>
<td>0.20</td>
<td>95</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>5.05</td>
<td>0.0010</td>
<td>5.04</td>
<td>4.93</td>
<td>5.06</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>3.00</td>
<td>0.0010</td>
<td>3.00</td>
<td>2.85</td>
<td>2.99</td>
</tr>
<tr>
<td>0.20</td>
<td>105</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>1.65</td>
<td>0.0009</td>
<td>1.67</td>
<td>1.53</td>
<td>1.64</td>
</tr>
<tr>
<td>0.20</td>
<td>110</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>0.85</td>
<td>0.0009</td>
<td>0.88</td>
<td>0.76</td>
<td>0.85</td>
</tr>
<tr>
<td>0.20</td>
<td>115</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>0.41</td>
<td>0.0008</td>
<td>0.44</td>
<td>0.36</td>
<td>0.41</td>
</tr>
<tr>
<td>0.20</td>
<td>120</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>0.19</td>
<td>0.0007</td>
<td>0.21</td>
<td>0.16</td>
<td>0.19</td>
</tr>
<tr>
<td>0.20</td>
<td>125</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>0.08</td>
<td>0.0005</td>
<td>0.10</td>
<td>0.07</td>
<td>0.09</td>
</tr>
<tr>
<td>0.20</td>
<td>130</td>
<td>12</td>
<td>0.12</td>
<td>1</td>
<td>0.03</td>
<td>0.0004</td>
<td>0.04</td>
<td>0.03</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 6.2: Option prices for different strike prices and stock dividends. MP: Milevsky and Posner. MP c: MP with continuous averaging. MP2: MP with cumulant correction à la Turnbull and Wakeman.
Figure 6.1: Option prices less Monte Carlo estimates. MP: Milevsky and Posner. MP2: MP with cumulant correction à la Turnbull and Wakeman.
Figure 6.2: Top: Option prices less Monte Carlo estimates. MP: Milevsky and Posner. MP2: MP with cumulant correction à la Turnbull and Wakeman. Bottom: Difference between the lognormal density and the reciprocal gamma density for the same mean and variance.
Figure 6.3: Top: Milevsky and Posner option prices less Monte Carlo estimates for different stock dividends $q$. Bottom: Difference between Milevsky and Posner option prices with discrete and continuous averaging. MP: Milevsky and Posner. MP2: MP with cumulant correction à la Turnbull and Wakeman.
\[ \sigma \quad K \quad n \quad T \quad MP \quad MP \ c \quad MP2 \]

\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
0.20 & 100 & 12 & 1 & 6.13 & 5.75 & 6.16 \\
0.20 & 100 & 24 & 1 & 5.94 & 5.75 & 5.97 \\
0.20 & 100 & 36 & 1 & 5.88 & 5.75 & 5.90 \\
0.20 & 100 & 48 & 1 & 5.84 & 5.75 & 5.87 \\
0.20 & 100 & 60 & 1 & 5.82 & 5.75 & 5.85 \\
0.20 & 100 & 72 & 1 & 5.81 & 5.75 & 5.83 \\
0.20 & 100 & 84 & 1 & 5.80 & 5.75 & 5.82 \\
0.20 & 100 & 96 & 1 & 5.80 & 5.75 & 5.82 \\
0.20 & 100 & 108 & 1 & 5.79 & 5.75 & 5.81 \\
0.20 & 100 & 120 & 1 & 5.79 & 5.75 & 5.81 \\
\hline
\end{tabular}

Table 6.3: Option prices for different numbers of prices in the average. MP: Milevsky and Posner. MP c: MP with continuous averaging. MP2: MP with cumulant correction à la Turnbull and Wakeman.

### 6.3.1 Conclusion

Our results show no indication that the reciprocal gamma distribution is a better approximation to the distribution of the arithmetic stock price average than the lognormal distribution. The pricing errors of the Milevsky and Posner method and the Levy method seem to be negatively correlated, so we suggested that taking the average of the prices would improve the accuracy, which was supported by numerical results.

[Milevsky and Posner, 1998] state that the pricing method should perform at its best when \( q > r \), but our results show no noteworthy increase in accuracy for options with high stock dividends. Increasing \( q \) for fixed \( K \) implies that the option’s probability of finishing in-the-money is decreasing, which explains the shift seen in Figure 6.3.

We improved the Milevsky and Posner method to take differences in skewness and kurtosis into account but, as in the Turnbull and Wakeman case, the pricing errors got out of control for long maturities. Furthermore, this approach was dominated by the TW algorithm for most values of the strike price.

Finally, we concluded that even though the discrete average converge to the continuous average, there are substantial price differences for small numbers of prices in the average. Since all traded Asian options are based on a discrete average, this implies that continuous averaging models should be used with caution.
CHAPTER 6. MILEVSKY AND POSNER METHOD
Chapter 7

Vorst method

[Vorst, 1992] gives an exact pricing formula for Asian options based on a geometric average. He uses this to approximate the arithmetic Asian option price by adjusting the strike price with the difference in expectation of the arithmetic and geometric averages. Furthermore, he obtains upper and lower bounds on the price.

7.1 Method

7.1.1 Model

The stock price is given by

\[ S_t = S_0 \exp((r - q - \frac{1}{2}\sigma^2)t + \sigma W_t) \]  

(7.1)

The geometric average is

\[ G = (\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} \]  

(7.2)

and the arithmetic average is

\[ A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i} \]  

(7.3)

where the averaging times as usual are equidistant

\[ t_i = i h, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n \]  

(7.4)
7.1.2 Geometric Asian option

From (7.2) the natural logarithm of the geometric average is

\[
\ln G = \frac{1}{n} \sum_{i=1}^{n} \ln S_{t_i} = \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} \ln \frac{S_{t_i}}{S_0} \tag{7.5}
\]

(7.1) gives that

\[
\ln \frac{S_{t_i}}{S_0} = (r - q - \frac{1}{2} \sigma^2) t_i + \sigma W_{t_i}
\]

\[
\sim N((r - q - \frac{1}{2} \sigma^2) t_i, \sigma^2 t_i), \quad i = 1, 2, \ldots, n \tag{7.6}
\]

Since the Brownian Motion has independent increments, we conclude from (7.5) and (7.6) that \( \ln G \) is normally distributed and we shall now find the mean \( \mu_G \) and the variance \( \sigma_G^2 \).

The mean of \( \ln G \) is

\[
\mu_G = \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} E[\ln \frac{S_{t_i}}{S_0}]
\]

\[
= \ln S_0 + \frac{1}{n} \sum_{i=1}^{n} (r - q - \frac{1}{2} \sigma^2) t_i
\]

\[
= \ln S_0 + \frac{1}{n} (r - q - \frac{1}{2} \sigma^2) \sum_{i=1}^{n} i h
\]

\[
= \ln S_0 + \frac{1}{n} (r - q - \frac{1}{2} \sigma^2) h \frac{n + 1}{2} \frac{n}{n}
\]

\[
= \ln S_0 + (r - q - \frac{1}{2} \sigma^2) T + \frac{h}{2} \tag{7.7}
\]

where we have used the summation formula

\[
\sum_{i=1}^{n} i = \frac{n + 1}{2} \frac{n}{n} \tag{7.8}
\]
The variance of \( \ln G \) is

\[
\sigma^2_{\ln G} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(\ln \frac{S_i}{S_0}, \ln \frac{S_j}{S_0})
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma^2 Cov(W_{i,}, W_{j,})
\]

\[
= \frac{\sigma^2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min(t_i, t_j)
\]

\[
= \frac{\sigma^2 h}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min(i, j) \quad (7.9)
\]

To calculate the double sum we use the summation formula (7.8) and

\[
\sum_{i=1}^{n} i^2 = \frac{(2n + 1)(n + 1)n}{6} \quad (7.10)
\]

We get

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \min(i, j) = \sum_{i=1}^{n} \sum_{j=1}^{i} \min(i, j) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \min(i, j)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{i} j + \sum_{i=1}^{n} \sum_{j=i+1}^{n} i
\]

\[
= \sum_{i=1}^{n} \frac{i+1}{2} i + \sum_{i=1}^{n} (n-i)i
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} (i^2 + i + 2ni - 2i^2)
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} ((2n+1)i - i^2)
\]

\[
= \frac{2n + 1}{2} \sum_{i=1}^{n} i - \frac{1}{2} \sum_{i=1}^{n} i^2
\]

\[
= \frac{2n + 1}{2} \frac{n+1}{2} n - \frac{1}{2} \frac{(2n + 1)(n + 1)n}{6}
\]

\[
= \frac{(2n + 1)(n + 1)n}{6} \quad (7.11)
\]
The variance can now be expressed as
\[
\sigma_G^2 = \frac{\sigma^2 h}{n^2} \sum_{i=1}^n \sum_{j=1}^n \min(i, j)
\]
\[
= \sigma^2 h \frac{(2n + 1)(n + 1) n}{6 n^2}
\]
\[
= \sigma^2 h \frac{(2n + 1)(n + 1)}{6 n} \quad (7.12)
\]
For easier comparison with [Vorst, 1992], we can also write
\[
\sigma_G^2 = \sigma^2 h \frac{(2n - 1)(n - 1) + 6 n}{6 n}
\]
\[
= \sigma^2 (h(n - 1) \frac{(2n - 1)}{6 n} + h)
\]
\[
= \sigma^2 (h + (T - h) \frac{(2n - 1)}{6 n}) \quad (7.13)
\]
We have seen that \( \ln G \sim N(\mu_G, \sigma_G^2) \), so the geometric average \( G \) is lognormally distributed and we are now able to price the option based on the geometric average.

**Theorem 13** The price at time 0 of the geometric Asian option with strike \( K \) and maturity \( T \) is
\[
C_G = \exp(-rT) \{ \exp(\mu_G + \frac{1}{2} \sigma_G^2) \Phi(d_1) - K \Phi(d_2) \} \quad (7.14)
\]
where
\[
d_1 = \frac{\mu_G - \ln K + \sigma_G^2}{\sigma_G} \quad (7.15)
\]
\[
d_2 = d_1 - \sigma_G \quad (7.16)
\]
**Proof.** The payoff of the option is \( \text{Max}(G - K, 0) \) so the price is
\[
C_G = \exp(-rT) \mathbb{E}[(G - K)^+]
\]
\[
= \exp(-rT) \int_{K}^{\infty} (x - K) g(x) dx
\]
where \( g \) is the lognormal density function of \( G \). The result now follows from the proof of Theorem 8. ■
7.1. METHOD

7.1.3 Arithmetic option price and bounds

Since the geometric average is less than the arithmetic average, \( G \leq A \), the geometric option price is a lower bound for the arithmetic option price

\[
C_G = \exp(-rT)E[(G - K)^+] \leq \exp(-rT)E[(A - K)^+] = C \quad (7.17)
\]

The inequality

\[
\max(A - K, 0) = \max(G - K, G - A) + A - G \\
\leq \max(G - K, 0) + A - G \quad (7.18)
\]

leads to the upper bound

\[
\exp(-rT)E[\max(G - K, 0) + A - G] \\
= \exp(-rT)\{E[(G - K)^+] + E[A] - E[G]\} \\
= C_G + \exp(-rT)(E[A] - E[G]) \quad (7.19)
\]

Thus, we have obtained the following bounds on the arithmetic option price

\[
C_G \leq C \leq C_G + \exp(-rT)(E[A] - E[G]) \quad (7.20)
\]

To calculate the upper bound we must calculate the mean of the arithmetic average. From (7.1), (7.3), (7.6) and (5.33)

\[
E[A] = E\left[\frac{1}{n} \sum_{i=1}^{n} S_{t_i}\right] \\
= \frac{S_0}{n} \sum_{i=1}^{n} E\left[\frac{S_{t_i}}{S_0}\right] \\
= \frac{S_0}{n} \sum_{i=1}^{n} \exp((r - q - \frac{1}{2}\sigma^2)t_i + \frac{1}{2}\sigma^2 t_i) \\
= \frac{S_0}{n} \sum_{i=1}^{n} \exp((r - q)i\Delta) \\
= \frac{S_0}{n} \frac{e^{(r-q)h} - e^{(r-q)(n+1)h}}{1 - e^{(r-q)\Delta}} \\
= \frac{S_0}{n} \frac{e^{(r-q)h} [1 - e^{(r-q)nh}]}{1 - e^{(r-q)\Delta}} \quad (7.21)
\]

where we have used the summation formula

\[
\sum_{i=1}^{n} e^{ai} = \frac{e^{a} - e^{a(n+1)}}{1 - e^{a}}, \quad a \neq 0 \quad (7.22)
\]
Since $G$ is lognormally distributed with parameters $\mu_G$ and $\sigma_G^2$, the mean of the geometric average is

$$E[G] = \exp(\mu_G + \frac{1}{2}\sigma_G^2)$$ (7.23)

Once it has been established that a price is in a certain interval, it is tempting to approximate the price with the value of the interval midpoint. [Vorst, 1992] takes another approach since he approximates the arithmetic option price with the price of a geometric option with a downward adjusted strike. The rationale is that the geometric and arithmetic averages are strongly correlated and so are the prices of the corresponding options. Moreover, pricing with a lower strike will to some extent compensate for the fact that the geometric option price is always lower than the arithmetic one.

**Theorem 14** Define the Vorst price at time 0 of the arithmetic Asian option with strike $K$ and maturity $T$ as

$$\tilde{C} = \exp(-rT)E[(G - K')^+]$$ (7.24)

where the adjusted strike price is

$$K' = K - (E[A] - E[G])$$ (7.25)

The pricing formula is then

$$\tilde{C} = \exp(-rT)\{\exp(\mu_G + \frac{1}{2}\sigma_G^2)\Phi(d'_1) - K'\Phi(d'_2)\}$$ (7.26)

where $\mu_G$ and $\sigma_G^2$ are obtained from (7.7) and (7.12) and

$$d'_1 = \frac{\mu_G - \ln K' + \sigma_G^2}{\sigma_G}$$ (7.27)

$$d'_2 = d'_1 - \sigma_G$$ (7.28)

The approximate option price $\tilde{C}$ is also in between the upper and lower bounds in (7.20), which implies that the pricing error is bounded by

$$|\tilde{C} - C| \leq \exp(-rT)(E[A] - E[G])$$ (7.29)

**Proof.** (7.26) follows directly from applying Theorem 13 to the geometric Asian option with strike $K'$, cf. (7.25). We have to show that the price $\tilde{C}$ is in between the bounds in (7.20). Since

$$\max(G - K, 0) \leq \max(G - K + E[A] - E[G], 0)$$
$$\leq \max(G - K, 0) + E[A] - E[G]$$
it follows that

\[
C_G = \exp(-rT)E[(G - K)^+]
\]
\[
\leq \exp(-rT)E[(G - K + E[A] - E[G])^+]
\]
\[
= \hat{C}
\]
\[
\leq \exp(-rT)E[(G - K)^+ + E[A] - E[G]]
\]
\[
= C_G + \exp(-rT)(E[A] - E[G])
\]

The same bounds apply for the true price \( C \), cf. (7.20), so we conclude that the absolute difference between \( \hat{C} \) and \( C \) is less than the difference between the upper bound and the lower bound

\[
|\hat{C} - C| \leq C_G + \exp(-rT)(E[A] - E[G]) - C_G
\]
\[
= \exp(-rT)(E[A] - E[G])
\]

\[\blacksquare\]

### 7.1.4 Method overview

The Vorst method can be summarized as follows

- Calculate the difference of the mean of the arithmetic and geometric averages.
- Reduce the strike price by the difference above.
- Use Theorem 14 to calculate the price of the arithmetic Asian option.
- Obtain the pricing bounds in (7.20) on the arithmetic Asian option by calculating the price of the geometric Asian option in Theorem 13.

### 7.2 Numerical results

Table 7.1 contains option prices for different values of stock price volatility \( \sigma \), strike price \( K \), number of prices in the average \( n \) and maturity \( T \). The Monte Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

\[
q = 0, \quad r = 0.05, \quad S_0 = 100
\]

(7.30)
CHAPTER 7. VORST METHOD

We see that the bounds labelled \( a_p \) and \( C_G \) are very poor in the sense that they are far away from the Monte Carlo estimate. Figure 7.1 shows that the accuracy of the Vorst method deteriorates as volatility and maturity increase. It appears from Figure 7.2 that the Vorst method slightly overprices in-the-money options while at-the-money and out-of-the-money options are heavily under-priced. When we think of it, the substantial pricing errors should not come as a big surprise, since we approximate the arithmetic average with a lognormal distribution and only correct for the mean.

The table and the figures in this section have also been used to illustrate two points from the previous chapter. The first point is that taking the average of the Levy price and the Milevsky and Posner price (MPLevy/MP+L) leads to a superior accuracy compared to all the other methods we have considered so far. The second point is that the generalized Edgeworth series expansion is better suited for a lognormal distribution (TW) than for a reciprocal gamma distribution (MP2).

### 7.2.1 Conclusion

The Vorst method is the first method in this thesis to establish bounds on the option price. We saw that the same bounds applied to both the true price and the approximation but unfortunately they did not constrain the prices enough to be useful. Neither were the Vorst prices very encouraging but all our efforts have not been in vain. We used the geometric average to price options based on an arithmetic average but, as we shall see in the next chapter, we far from used all its potential.
### 7.2. NUMERICAL RESULTS

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( K )</th>
<th>( n )</th>
<th>( T )</th>
<th>MC</th>
<th>sd</th>
<th>Vorst</th>
<th>up</th>
<th>( C_G )</th>
<th>MP+L</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>2.93</td>
<td>0.0001</td>
<td>2.93</td>
<td>2.93</td>
<td>2.90</td>
<td>2.93</td>
</tr>
<tr>
<td>0.10</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>3.90</td>
<td>0.0004</td>
<td>3.90</td>
<td>3.93</td>
<td>3.84</td>
<td>3.90</td>
</tr>
<tr>
<td>0.15</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>5.01</td>
<td>0.0009</td>
<td>5.00</td>
<td>5.07</td>
<td>4.88</td>
<td>5.01</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.13</td>
<td>6.27</td>
<td>5.94</td>
<td>6.15</td>
</tr>
<tr>
<td>0.25</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>7.31</td>
<td>0.0024</td>
<td>7.27</td>
<td>7.51</td>
<td>6.99</td>
<td>7.31</td>
</tr>
<tr>
<td>0.30</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>8.47</td>
<td>0.0035</td>
<td>8.40</td>
<td>8.76</td>
<td>8.02</td>
<td>8.47</td>
</tr>
<tr>
<td>0.35</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>9.64</td>
<td>0.0049</td>
<td>9.53</td>
<td>10.03</td>
<td>9.04</td>
<td>9.63</td>
</tr>
<tr>
<td>0.40</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>10.80</td>
<td>0.0065</td>
<td>10.66</td>
<td>11.32</td>
<td>10.03</td>
<td>10.79</td>
</tr>
<tr>
<td>0.45</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>11.96</td>
<td>0.0085</td>
<td>11.77</td>
<td>12.63</td>
<td>11.00</td>
<td>11.94</td>
</tr>
<tr>
<td>0.50</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>13.11</td>
<td>0.0107</td>
<td>12.86</td>
<td>13.95</td>
<td>11.94</td>
<td>13.09</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.5</td>
<td>4.11</td>
<td>0.0007</td>
<td>4.10</td>
<td>4.17</td>
<td>4.01</td>
<td>4.11</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.0</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.13</td>
<td>6.27</td>
<td>5.94</td>
<td>6.15</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.5</td>
<td>7.84</td>
<td>0.0025</td>
<td>7.78</td>
<td>8.00</td>
<td>7.50</td>
<td>7.83</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.0</td>
<td>9.32</td>
<td>0.0034</td>
<td>9.24</td>
<td>9.52</td>
<td>8.85</td>
<td>9.31</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.5</td>
<td>10.66</td>
<td>0.0045</td>
<td>10.55</td>
<td>10.89</td>
<td>10.05</td>
<td>10.65</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.0</td>
<td>11.90</td>
<td>0.0055</td>
<td>11.76</td>
<td>12.16</td>
<td>11.15</td>
<td>11.89</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.5</td>
<td>13.05</td>
<td>0.0066</td>
<td>12.88</td>
<td>13.34</td>
<td>12.16</td>
<td>13.04</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.0</td>
<td>14.14</td>
<td>0.0078</td>
<td>13.93</td>
<td>14.45</td>
<td>13.09</td>
<td>14.12</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.5</td>
<td>15.16</td>
<td>0.0090</td>
<td>14.91</td>
<td>15.49</td>
<td>13.96</td>
<td>15.14</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>5.0</td>
<td>16.12</td>
<td>0.0102</td>
<td>15.85</td>
<td>16.47</td>
<td>14.77</td>
<td>16.10</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>6.0</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.13</td>
<td>6.27</td>
<td>5.94</td>
<td>6.15</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>24</td>
<td>1</td>
<td>5.96</td>
<td>0.0016</td>
<td>5.93</td>
<td>6.08</td>
<td>5.74</td>
<td>5.96</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>36</td>
<td>1</td>
<td>5.90</td>
<td>0.0016</td>
<td>5.86</td>
<td>6.01</td>
<td>5.68</td>
<td>5.89</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>48</td>
<td>1</td>
<td>5.86</td>
<td>0.0016</td>
<td>5.83</td>
<td>5.98</td>
<td>5.64</td>
<td>5.86</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>60</td>
<td>1</td>
<td>5.84</td>
<td>0.0016</td>
<td>5.81</td>
<td>5.96</td>
<td>5.63</td>
<td>5.84</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>70</td>
<td>1</td>
<td>31.10</td>
<td>0.0015</td>
<td>31.16</td>
<td>31.16</td>
<td>30.83</td>
<td>31.16</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>75</td>
<td>1</td>
<td>26.42</td>
<td>0.0015</td>
<td>26.42</td>
<td>26.42</td>
<td>26.09</td>
<td>26.42</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>80</td>
<td>1</td>
<td>21.72</td>
<td>0.0015</td>
<td>21.72</td>
<td>21.73</td>
<td>21.40</td>
<td>21.72</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>85</td>
<td>1</td>
<td>17.16</td>
<td>0.0015</td>
<td>17.17</td>
<td>17.20</td>
<td>16.86</td>
<td>17.16</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>90</td>
<td>1</td>
<td>12.92</td>
<td>0.0015</td>
<td>12.92</td>
<td>12.98</td>
<td>12.64</td>
<td>12.92</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>95</td>
<td>1</td>
<td>9.19</td>
<td>0.0015</td>
<td>9.18</td>
<td>9.28</td>
<td>8.95</td>
<td>9.19</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.13</td>
<td>6.27</td>
<td>5.94</td>
<td>6.15</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>105</td>
<td>1</td>
<td>3.87</td>
<td>0.0016</td>
<td>3.82</td>
<td>4.02</td>
<td>3.69</td>
<td>3.87</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>110</td>
<td>1</td>
<td>2.29</td>
<td>0.0015</td>
<td>2.23</td>
<td>2.48</td>
<td>2.14</td>
<td>2.29</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>115</td>
<td>1</td>
<td>1.28</td>
<td>0.0015</td>
<td>1.22</td>
<td>1.50</td>
<td>1.17</td>
<td>1.28</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>120</td>
<td>1</td>
<td>0.67</td>
<td>0.0013</td>
<td>0.63</td>
<td>0.93</td>
<td>0.60</td>
<td>0.68</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>125</td>
<td>1</td>
<td>0.34</td>
<td>0.0012</td>
<td>0.30</td>
<td>0.62</td>
<td>0.29</td>
<td>0.34</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>130</td>
<td>1</td>
<td>0.16</td>
<td>0.0010</td>
<td>0.14</td>
<td>0.47</td>
<td>0.13</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 7.1: Option prices and bounds for different volatilities, maturities, strike prices and number of prices in the average. up: Vorst upper bound. \( C_G \): Lower bound (geometric option price). MP+L: Average of Levy price and Milevsky and Posner price.
Figure 7.1: Vorst option prices less Monte Carlo estimates. TW, MPLevy and MP2 correspond to methods treated in previous chapters.
7.2. NUMERICAL RESULTS

Figure 7.2: Vorst option prices less Monte Carlo estimates. TW, MPLevy and MP2 correspond to methods treated in previous chapters.
Chapter 8

Curran method

[Curran, 1994] introduces an extremely precise method for pricing Asian options and options on portfolios. The approach is to condition on the geometric average and integrate with respect to its lognormal distribution. Curran develops two different approximations, and we shall use the so-called naive approximation to obtain a lower bound on the Asian option price.

8.1 Method

8.1.1 Model

The stock price follows the stochastic differential equation

\[ dS_t = (r - q)S_t dt + \sigma S_t dW_t \]  

(8.1)

and the arithmetic average of the stock prices is

\[ A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i} \]  

(8.2)

where the time points are equidistant

\[ t_i = ih, \quad h = \frac{T}{n}, \quad i = 1, 2, ..., n \]  

(8.3)

(8.1) implies that the stock price at time \( t_i \) is lognormally distributed, so

\[ \ln S_{t_i} = \ln S_0 + (r - q - \frac{1}{2} \sigma^2) t_i + \sigma W_{t_i} \]

\[ \sim N(\mu_i, \sigma^2_i), \quad i = 1, 2, ..., n \]  

(8.4)
where
\[
\mu_i = \ln S_0 + (r - q - \frac{1}{2}\sigma^2)t_i, \quad i = 1, 2, \ldots, n \tag{8.5}
\]
\[
\sigma_i^2 = \sigma^2 t_i, \quad i = 1, 2, \ldots, n \tag{8.6}
\]
The geometric average is
\[
G = \left( \prod_{i=1}^{n} S_{t_i} \right)^{\frac{1}{n}} \tag{8.7}
\]
and taking the natural logarithm leads to
\[
\ln G = \frac{1}{n} \sum_{i=1}^{n} \ln S_{t_i} \tag{8.8}
\]
From Chapter 7 we know that \(\ln G\) is normally distributed with parameters \(\mu_G\) and \(\sigma_G^2\) given by
\[
\mu_G = \ln S_0 + (r - q - \frac{1}{2}\sigma^2)\frac{T + h}{2} \tag{8.9}
\]
\[
\sigma_G^2 = \frac{\sigma^2 h (2n + 1)(n + 1)}{6n} \tag{8.10}
\]
To sum up, the marginal distributions of \(\ln S_t\) and \(\ln G\) are normal, and it turns out that they also have a jointly normal distribution.

**Proposition 15** The random vector \(\eta_i = (\ln S_{t_i}, \ln G)\) has a 2-dimensional normal distribution for \(i = 1, 2, \ldots, n\) with mean vector
\[
(\mu_i, \mu_G), \quad i = 1, 2, \ldots, n \tag{8.11}
\]
and covariance matrix
\[
\begin{bmatrix}
\sigma_i^2 & \gamma_i \\
\gamma_i & \sigma_G^2
\end{bmatrix}, \quad i = 1, 2, \ldots, n \tag{8.12}
\]
where
\[
\gamma_i = \text{Cov}(\ln S_{t_i}, \ln G) = \frac{\sigma^2 h ((2n + 1)i - i^2)}{2n}, \quad i = 1, 2, \ldots, n \tag{8.13}
\]
**Proof.** Let \(i \in \{1, 2, \ldots, n\}\). We can split the Wiener process at time \(t_n\) into independent increments
\[
W_{t_n} = (W_{t_n} - W_{t_{n-1}}) + (W_{t_{n-1}} - W_{t_{n-2}}) + \ldots + (W_{t_1} - W_0) + W_0
\]
\[
= \Delta W_{t_n} + \Delta W_{t_{n-1}} + \ldots + \Delta W_{t_1}
\]
8.1. METHOD

Define the vector of increments

\[ W^\Delta = (\Delta W_{t_1}, \Delta W_{t_2}, ..., \Delta W_{t_n}) \]

which has a \( n \)-dimensional normal distribution. Since \( \ln S_{t_i} \) and \( \ln G \) are linear combinations of these increments, we can find a vector \( \xi_i \) and a matrix \( \Sigma_i \), so

\[ \eta_i = \xi_i + W^\Delta \Sigma_i \]

It follows from (4.22.16) in [Hoffmann-Jørgensen, 1994a] that the distribution of \( \eta_i \) is jointly normal. The covariance between \( \ln S_{t_i} \) and \( \ln G \) is

\[
\gamma_i = \text{Cov} (\ln S_{t_i}, \ln G) = \text{Cov} (\ln S_{t_i}, \frac{1}{n} \sum_{j=1}^{n} \ln S_{t_j}) \\
= \frac{1}{n} \sum_{j=1}^{n} \text{Cov} (\ln S_{t_i}, \ln S_{t_j}) = \frac{1}{n} \sum_{j=1}^{n} \sigma^2 \min(t_i, t_j) \\
= \frac{\sigma^2}{n} \sum_{j=1}^{n} \min(j, j) = \frac{\sigma^2}{n} \left( \sum_{j=1}^{i} \min(i, j) + \sum_{j=i+1}^{n} \min(i, j) \right) \\
= \frac{\sigma^2}{n} \left( \sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i \right) = \frac{\sigma^2}{n} \left( \frac{i+1}{2} \frac{n-1}{2} + (n-i) i \right) \\
= \frac{\sigma^2}{n} \left( ni + \frac{1}{2} i^2 - \frac{1}{2} i^2 \right) = \frac{\sigma^2}{2n} \left( (2n+1)i - i^2 \right)
\]

The next proposition shows that the normal distribution is stable to conditioning when the random variables are jointly normal.

**Proposition 16** The conditional distribution of \( \ln S_{t_i} \) given \( \ln G \) is a normal distribution

\[
(\ln S_{t_i} | \ln G = y) \sim N(\mu_i + (y - \mu_G) \frac{\gamma_i}{\sigma_G^2}, \sigma_i^2 - \frac{\gamma_i^2}{\sigma_G^2}), \quad i = 1, 2, ..., n \tag{8.14}
\]

Moreover, \( (S_{t_i} | \ln G) \) has a lognormal distribution and

\[
E[S_{t_i} | \ln G = y] = \exp(\mu_i + (y - \mu_G) \frac{\gamma_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\gamma_i^2}{\sigma_G^2})), \quad i = 1, 2, ..., n \tag{8.15}
\]
Proof. We conclude from (10.7.6) in [Hoffmann-Jørgensen, 1994b] and Proposition 15 that \((\ln S_t | \ln G)\) is normally distributed. The calculation of the mean and the variance in (8.14) is straightforward, but has been left out. It follows per definition that \((S_t | \ln G)\) is lognormally distributed with the same parameters, and the conditional expectation (8.15) is then calculated from (5.33).

8.1.2 Bounds on the option price

We condition on the geometric average \(G\) in the pricing expression of the Asian option

\[
C = e^{-rT} E[(A - K)^+] \\
= e^{-rT} E[E[(A - K)^+ | G]] \\
= e^{-rT} \int_0^\infty E[(A - K)^+ | G = x] g(x) dx
\]

(8.16)

where \(g\) is the lognormal density function of \(G\). Define

\[
C_1 = \int_0^K E[(A - K)^+ | G = x] g(x) dx
\]

(8.17)

\[
C_2 = \int_K^\infty E[(A - K)^+ | G = x] g(x) dx
\]

(8.18)

so

\[
C = e^{-rT} (C_1 + C_2)
\]

(8.19)

Since the arithmetic average is greater than the geometric average \(A \geq G\)

\[
C_2 = \int_K^\infty E[A - K | G = x] g(x) dx
\]

(8.20)

Loosely speaking, \(C_1\) corresponds to the situation where the geometric option finishes out-of-the-money and the status of the arithmetic option is unknown. Likewise, \(C_2\) corresponds to the situation where both options finish in-the-money. Proposition 16 enables us to calculate \(C_2\) from (8.20), but we have to settle for an approximation of \(C_1\). Using Jensen’s inequality we obtain a
8.1. METHOD

lower bound on $C_1$

$$C_1 = \int_0^K E[(A - K)^+ | G = x] g(x) dx$$
$$\geq \int_0^K (E[A - K | G = x])^+ g(x) dx$$
$$= \int_L E[A - K | G = x] g(x) dx$$
$$= \hat{C}_1$$  \hfill (8.21)

where

$$L = \{x | E[A|G = x] = K\}$$  \hfill (8.22)

To find $L$ we calculate the conditional expectation

$$E[A|G = x] = E[\frac{1}{n} \sum_{i=1}^n S_i|G = x] = \frac{1}{n} \sum_{i=1}^n E[S_i|\ln G = \ln x]$$
$$= \frac{1}{n} \sum_{i=1}^n \exp(\mu_i + (\ln x - \mu_G) \frac{\gamma_i}{\sigma_G} + \frac{1}{2} (\sigma_i^2 - \frac{\gamma_i^2}{\sigma_G}))$$  \hfill (8.23)

where we have used (8.15). Setting the last expression equal to $K$ and solving for $x$ is difficult, so we have applied a simple numerical method, which appears from the Visual Basic code in Appendix E. It is important that the approximation $\hat{L}$ is greater than $L$ to make sure that $\hat{C}_1$ is a lower bound on $C_1$.

**Theorem 17** A lower bound on the price of the Asian option with strike $K$ and maturity $T$ is

$$\hat{C} = \exp(-rT)(\hat{C}_1 + C_2)$$
$$= \exp(-rT)\{\frac{1}{n} \sum_{i=1}^n \exp(\mu_i + \frac{1}{2} \sigma_i^2) \Phi(\frac{\mu_G - \ln L + \gamma_i}{\sigma_G})$$
$$- K \Phi(\frac{\mu_G - \ln L}{\sigma_G})\}$$  \hfill (8.24)

where the relevant parameters are given in (8.5), (8.6), (8.9), (8.10), (8.13) and (8.22).
**Proof.** Collecting (8.21) and (8.20) gives

\[
\tilde{C}_1 + C_2 = \int_L^\infty E[A - K|G = x]g(x)dx \\
= \int_L^\infty E[A|G = x]g(x)dx - K\int_L^\infty g(x)dx \\
= \int_L^\infty E\left[\frac{1}{n}\sum_{i=1}^n S_{i|}G = x\right]g(x)dx - K\int_L^\infty g(x)dx \\
= \int_L^\infty \frac{1}{n}\sum_{i=1}^n E[S_{i|}G = x]g(x)dx - K\int_L^\infty g(x)dx \\
= \frac{1}{n}\sum_{i=1}^n \int_L^\infty E[S_{i|}\ln G = \ln x]g(x)dx - K\int_L^\infty g(x)dx
\]

We know from the proof of Theorem 8 that

\[
K\int_L^\infty g(x)dx = K\Phi\left(\frac{\mu_G - \ln L}{\sigma_G}\right)
\]

and from (8.15)

\[
\int_L^\infty E[S_{i|}\ln G = \ln x]g(x)dx \\
= \int_L^\infty \exp(\mu_i + (\ln x - \mu_G)\frac{\gamma_i}{\sigma_G} + \frac{1}{2}(\sigma_i^2 - \frac{\gamma_i^2}{\sigma_G^2}))g(x)dx \\
= \exp(\mu_i + \frac{1}{2}\sigma_i^2)\int_L^\infty \exp((\ln x - \mu_G)\frac{\gamma_i}{\sigma_G} - \frac{1}{2}\frac{\gamma_i^2}{\sigma_G^2})g(x)dx
\]

Using the density (5.38) of the lognormal distribution the last integral is

\[
\int_L^\infty \frac{1}{\sqrt{2\pi}\sigma_Gx}\exp((\ln x - \mu_G)\frac{\gamma_i}{\sigma_G} - \frac{1}{2}\frac{\gamma_i^2}{\sigma_G^2} - \frac{1}{2}\frac{\mu_G - \ln L}{\sigma_G^2})^2)dx
\]

We make the change of variables

\[
z = \frac{\mu_G - \ln x + \gamma_i}{\sigma_G}, \quad \frac{dz}{dx} = -\frac{1}{\sigma_Gx}
\]
8.1. METHOD

which leads to

\[
\frac{1}{\sqrt{2\pi}} \exp\left(\frac{\gamma_i}{\sigma_G^2} - \frac{1}{2} \frac{\gamma_i^2}{\sigma_G^2} - \frac{1}{2} (z - \gamma_i)^2 \right) dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \exp\left(-z \frac{\gamma_i}{\sigma_G} + \frac{1}{2} \frac{\gamma_i^2}{\sigma_G^2} - \frac{1}{2} z^2 - \frac{1}{2} \frac{\gamma_i^2}{\sigma_G^2} + z \gamma_i\right) dz
\]

\[
= \Phi\left(\frac{\mu_G - \ln L + \gamma_i}{\sigma_G}\right)
\]

Finally, we collect the terms

\[
\tilde{C}_1 + C_2 = \frac{1}{n} \sum_{i=1}^{n} \exp(\mu_i + \frac{1}{2} \sigma_i^2) \Phi\left(\frac{\mu_G - \ln L + \gamma_i}{\sigma_G}\right)
\]

from which (8.24) follows.  

[Curran, 1994] does not mention any upper bounds, but [Rogers and Shi, 1995], whose method relies on conditioning on the Brownian Motion, use the inequalities

\[
0 \leq E[U^+] - (E[U])^+ = \frac{1}{2} (E[U] - |E[U]|)
\]

\[
\leq \frac{1}{2} E[|U - E[U]|]
\]

\[
\leq \frac{1}{2} \sqrt{\text{Var}(U)}
\]

(8.25)

to form bounds similar to

\[
0 \leq E[E[(A - K)^+ | G] - (E[A - K | G])^+]
\]

\[
\leq \frac{1}{2} E[\sqrt{\text{Var}(A - K | G)}]
\]

(8.26)

From (8.21) an upper bound on \(C_1\) is then

\[
\tilde{C}_1 = \int_0^K (E[(A - K) | G = x])^+ g(x) dx
\]

\[
+ \frac{1}{2} \int_0^K \sqrt{\text{Var}(A - K | G)} g(x) dx
\]

(8.27)

Due to time constraints we have not been able to pursue this upper bound any further.
8.1.3 Method overview

The Curran method can be summarized as follows

- Calculate the mean and the variance of $\ln G$ (the logarithm of the geometric average).
- For each averaging time $t_i$ calculate the mean and the variance of $\ln S_{t_i}$ (the logarithm of the stock price at time $t_i$) and the covariance between $\ln G$ and $\ln S_{t_i}$.
- Use a numerical method to find the lower limit $L$ of the integral in (8.21).
- Obtain a lower bound on the price of the Asian option by means of Theorem 17.

8.2 Numerical results

Table 8.1 contains the Curran lower bound (CLB) on the option price for different values of stock price volatility $\sigma$, strike price $K$, number of prices in the average $n$ and maturity $T$. The Monte Carlo simulation has been carried out with 50,000 paths and the geometric option price as control variate, cf. Chapter 4. We have also included the approximation $\tilde{L}$ of the lower limit $L$ in the $C_1$ integral (8.21). Unless otherwise mentioned the stock dividend, the interest rate and the initial stock price have been fixed at

$$q = 0, \quad r = 0.05, \quad S_0 = 100 \quad (8.28)$$

The lower bound is extremely accurate, which we can see in the last column of Table 8.1. CLB and the Monte Carlo estimates only differ on the third or fourth decimal and CLB is inside the 95% confidence interval for all parameter values as seen in Figure 8.1 and Figure 8.2. These figures have also been used to illustrate that the Curran method generally is better than averaging the Levy price and the Milevsky and Posner price, cf. Chapter 6. Though, it is worth noting that MPLevy is inside the 95% confidence interval in most cases.

In Figure 8.2 we see that the lower bound is greater than the Monte Carlo estimate for most values of the strike price, which tells us two things. First of all the MC estimates are too low and secondly the Curran method is so accurate that even Monte Carlo simulation has problems matching it. We have run additional Monte Carlo simulations with 100,000 and 200,000
8.2. NUMERICAL RESULTS

paths, which show that CLB is below the MC estimates except when the option is deep-in-the-money. CLB is still inside the 95% confidence interval and the relative differences are of magnitude less than 0.01 %, which should not cause any concern.

Table 8.1 shows that \( \hat{L} \) is close to \( K \) and decreasing in \( \sigma \) and \( T \), which is in accordance with the differences between the geometric and arithmetic option prices seen in Figure 4.4 and Figure 4.5. The closer \( L \) (and hence \( \hat{L} \)) is to \( K \), the smaller \( \tilde{C}_1 \) is, which implies a more accurate lower bound \( \tilde{C} \). We also see that the difference between \( \hat{L} \) and \( K \) is smallest for \( K = 100 = S_0 \) and it increases as the strike price moves away from the initial stock price.

8.2.1 Conclusion

We implemented the lower bound in [Curran, 1994] on the price of an Asian option and the result was very inspiring. The lower bound is so precise that it is the true price for all practical purposes. In contrast to other methods, the accuracy is retained even for large volatilities and long maturities as well as the whole range of strike prices.

The Curran method revealed that our Monte Carlo estimates from simulating with 50,000 paths were slightly too small, since the estimates were below the lower bound in some cases. Increasing the number of paths to 200,000 fixed this, but was also very time-consuming.

The lower bound was inside the 95% confidence interval for the Monte Carlo estimates for all parameter values. To put this into perspective we used the simpler one of the two approximations in [Curran, 1994] since it does not require numerical integration. Finally, we suggested how to obtain an upper bound on the option price inspired by [Rogers and Shi, 1995].
<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>K</th>
<th>n</th>
<th>T</th>
<th>MC</th>
<th>sd</th>
<th>CLB</th>
<th>L</th>
<th>CLB-MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>2.93</td>
<td>0.0001</td>
<td>2.93</td>
<td>99.9856</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.10</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>3.90</td>
<td>0.0004</td>
<td>3.90</td>
<td>99.9456</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.15</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>5.01</td>
<td>0.0009</td>
<td>5.01</td>
<td>99.8791</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.16</td>
<td>99.7859</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.25</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>7.31</td>
<td>0.0024</td>
<td>7.31</td>
<td>99.6662</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.30</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>8.47</td>
<td>0.0035</td>
<td>8.47</td>
<td>99.5198</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.35</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>9.64</td>
<td>0.0049</td>
<td>9.64</td>
<td>99.3467</td>
<td>0.0002</td>
</tr>
<tr>
<td>0.40</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>10.80</td>
<td>0.0065</td>
<td>10.80</td>
<td>99.1473</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.45</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>11.96</td>
<td>0.0085</td>
<td>11.96</td>
<td>98.9212</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.50</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>13.11</td>
<td>0.0107</td>
<td>13.12</td>
<td>98.6684</td>
<td>0.0008</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>0.5</td>
<td>4.11</td>
<td>0.0007</td>
<td>4.11</td>
<td>99.8931</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.0</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.16</td>
<td>99.7859</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1.5</td>
<td>7.84</td>
<td>0.0025</td>
<td>7.84</td>
<td>99.6786</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.0</td>
<td>9.32</td>
<td>0.0034</td>
<td>9.32</td>
<td>99.5712</td>
<td>0.0002</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>2.5</td>
<td>10.66</td>
<td>0.0045</td>
<td>10.66</td>
<td>99.4637</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.0</td>
<td>11.90</td>
<td>0.0055</td>
<td>11.90</td>
<td>99.3561</td>
<td>0.0002</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>3.5</td>
<td>13.05</td>
<td>0.0066</td>
<td>13.05</td>
<td>99.2483</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.0</td>
<td>14.14</td>
<td>0.0078</td>
<td>14.14</td>
<td>99.1405</td>
<td>0.0007</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>4.5</td>
<td>15.16</td>
<td>0.0090</td>
<td>15.16</td>
<td>99.0325</td>
<td>0.0012</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>5.0</td>
<td>16.12</td>
<td>0.0102</td>
<td>16.12</td>
<td>98.9243</td>
<td>0.0019</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.16</td>
<td>99.7859</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>24</td>
<td>1</td>
<td>5.96</td>
<td>0.0016</td>
<td>5.96</td>
<td>99.7921</td>
<td>-0.0011</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>36</td>
<td>1</td>
<td>5.90</td>
<td>0.0016</td>
<td>5.89</td>
<td>99.7945</td>
<td>-0.0020</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>48</td>
<td>1</td>
<td>5.86</td>
<td>0.0016</td>
<td>5.86</td>
<td>99.7958</td>
<td>-0.0026</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>60</td>
<td>1</td>
<td>5.84</td>
<td>0.0016</td>
<td>5.84</td>
<td>99.7966</td>
<td>-0.0027</td>
</tr>
<tr>
<td>0.20</td>
<td>70</td>
<td>12</td>
<td>1</td>
<td>31.16</td>
<td>0.0015</td>
<td>31.16</td>
<td>69.1140</td>
<td>0.0009</td>
</tr>
<tr>
<td>0.20</td>
<td>75</td>
<td>12</td>
<td>1</td>
<td>26.42</td>
<td>0.0015</td>
<td>26.42</td>
<td>74.3369</td>
<td>0.0008</td>
</tr>
<tr>
<td>0.20</td>
<td>80</td>
<td>12</td>
<td>1</td>
<td>21.72</td>
<td>0.0015</td>
<td>21.72</td>
<td>79.5133</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.20</td>
<td>85</td>
<td>12</td>
<td>1</td>
<td>17.16</td>
<td>0.0015</td>
<td>17.16</td>
<td>84.6450</td>
<td>0.0006</td>
</tr>
<tr>
<td>0.20</td>
<td>90</td>
<td>12</td>
<td>1</td>
<td>12.92</td>
<td>0.0015</td>
<td>12.92</td>
<td>89.7334</td>
<td>0.0007</td>
</tr>
<tr>
<td>0.20</td>
<td>95</td>
<td>12</td>
<td>1</td>
<td>9.19</td>
<td>0.0015</td>
<td>9.19</td>
<td>94.7799</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.20</td>
<td>100</td>
<td>12</td>
<td>1</td>
<td>6.16</td>
<td>0.0016</td>
<td>6.16</td>
<td>99.7859</td>
<td>0.0003</td>
</tr>
<tr>
<td>0.20</td>
<td>105</td>
<td>12</td>
<td>1</td>
<td>3.87</td>
<td>0.0016</td>
<td>3.87</td>
<td>104.7530</td>
<td>0.0002</td>
</tr>
<tr>
<td>0.20</td>
<td>110</td>
<td>12</td>
<td>1</td>
<td>2.29</td>
<td>0.0015</td>
<td>2.29</td>
<td>109.6826</td>
<td>-0.0001</td>
</tr>
<tr>
<td>0.20</td>
<td>115</td>
<td>12</td>
<td>1</td>
<td>1.28</td>
<td>0.0015</td>
<td>1.28</td>
<td>114.5759</td>
<td>-0.0005</td>
</tr>
<tr>
<td>0.20</td>
<td>120</td>
<td>12</td>
<td>1</td>
<td>0.67</td>
<td>0.0013</td>
<td>0.67</td>
<td>119.4341</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.20</td>
<td>125</td>
<td>12</td>
<td>1</td>
<td>0.34</td>
<td>0.0012</td>
<td>0.34</td>
<td>124.2585</td>
<td>0.0007</td>
</tr>
<tr>
<td>0.20</td>
<td>130</td>
<td>12</td>
<td>1</td>
<td>0.16</td>
<td>0.0010</td>
<td>0.16</td>
<td>129.0501</td>
<td>0.0012</td>
</tr>
</tbody>
</table>

Table 8.1: Curran lower bound (CLB) on the Asian option price for different values of volatility, strike price, number of prices in the average and maturity. MC: Monte Carlo estimate. sd: Standard deviation of MC estimator. L: Approximation of L in (8.21).
Figure 8.1: Curran lower bound on the Asian option price and 95% confidence interval for the Monte Carlo estimate. MPLLevy: Average of Levy price and Milevsky and Posner price.
Figure 8.2: Curran lower bound on the Asian option price and 95% confidence interval for the Monte Carlo estimate. MPLevy: Average of Levy price and Milevsky and Posner price.
Chapter 9

Conclusion

Numerical results

In this thesis I have derived and implemented selected methods for pricing Asian options. First of all, I looked at Monte Carlo simulation with and without variance reduction methods. The numerical results led me to conclude that using the price of the geometric Asian option as control variate is by far the best way to price the arithmetic Asian option. This is due to the high correlation between the geometric average and the arithmetic average. The estimates of the standard method and the antithetic method had standard deviations so large that they could not be relied on, and reducing the standard deviations by increasing the number of paths was not computationally feasible. I also realized that the quality of the random numbers is very important and I expect that a more sophisticated random number generator can lead to better Monte Carlo estimates.

The pricing method in [Turnbull and Wakeman, 1991] performed very well for small stock price volatilities and short maturities. For large volatilities and maturities longer than a couple of years, the kurtosis correction in the generalized Edgeworth series expansion resulted in large price deviations from the Monte Carlo estimates. I studied the impact of removing correction terms in the pricing expression and I concluded that not adjusting for differences in skewness and kurtosis, which is equivalent to the method in [Levy, 1992], improved the accuracy in some cases.

Having used the lognormal distribution as approximation for the unknown distribution of the arithmetic average, I turned to the reciprocal gamma distribution suggested by [Milevsky and Posner, 1998]. The numerical results showed no improvement compared to the Levy method but the pricing errors of the two methods seemed to be negatively correlated. This gave me the idea to take the average of the prices, which turned out surprisingly well in terms

93
of accuracy. I also applied the generalized Edgeworth series expansion in the Milevsky and Posner setup, which led to good results for short maturities. As in the Turnbull and Wakeman case, the pricing errors were substantial for maturities longer than a couple of years, and the accuracy was better with the lognormal distribution in general. Finally, I showed a considerable price difference between Asian options based on discrete and continuous averages.

The lognormal approximation in [Vorst, 1992] was only acceptable for small volatilities and in-the-money options with short maturities. I also derived a pricing formula for the geometric Asian option, which was used as a lower bound on the price of the arithmetic Asian option. The pricing bounds were very easy to calculate but they were too wide to constrain the price sufficiently.

The last pricing method was the lower bound in [Curran, 1994]. The numerical results were almost indistinguishable from the Monte Carlo estimates, so I concluded that the lower bound is the true price for all practical purposes. Increasing the stock price volatility and the maturity did not spoil the accuracy, since the lower bound was still inside the 95% confidence interval for the Monte Carlo estimates. The Curran method was actually more precise than Monte Carlo simulation with 50,000 paths, and it took more than 100,000 paths to get the Monte Carlo estimates above the lower bound for most values of the strike price.

Comparison of pricing methods

I have compared the methods as they appeared in the previous chapters and I shall summarize the conclusions here. It is difficult to rank the methods in terms of accuracy since none of them are uniformly better than any other method. Furthermore, many parameters can be varied, so I shall draw conclusions on a general purpose basis.

The Vorst method heavily underpriced out-of-the-money options and consequently it is my candidate for the least usable method. Then follow the Levy method and the Milevsky and Posner method due to their large variation and sensitivity to volatility and maturity. Next in line are the Turnbull and Wakeman method and its reciprocal gamma equivalent MP2, since they performed well in many cases, but unfortunately they were next to useless for long maturities. My idea of taking the average of the Levy price and the Milevsky and Posner price is the second best method. For a wide range of parameter values, it gave prices within the 95% confidence interval for the Monte Carlo estimates. The best method, in my opinion, is the Curran method, because the lower bound was very close to the Monte Carlo estimates even for large values of volatility and maturity.
Thus, the main conclusions of this thesis are:

- The lower bound in [Curran, 1994] on the price of an arithmetic Asian option is very accurate and reliable.

- The price of the geometric Asian option is a very suitable control variate for Monte Carlo simulation of the price of an arithmetic Asian option.

**Computational aspects**

An in-depth discussion of the programming is not appropriate in my point of view, but I would like to add a few remarks. For anyone with some programming experience, all the methods should be easy to implement by means of this thesis. Extending the code to a more generalized setting should not pose a problem and neither should improvements to reduce memory usage and computational time. The priority for me has been to write code that is usable and reliable.

The Monte Carlo simulation is very time-consuming, especially when using many paths to obtain a high level of precision. I used a very simple numerical method in the implementation of the Curran method, so obtaining one bound takes a couple of seconds. The rest of the methods only use a fraction of a second per price, which implies that all methods except Monte Carlo simulation can be used in real time to support trading.

**Future research**

The basic Black-Scholes model used in this thesis enables many generalizations. It could be interesting to investigate if the conclusions made also apply to setups with e.g. pricing in the averaging period, flexible and floating Asian options, non-lognormal underlying assets and a stochastic term structure of interest rates.

Calculation of accurate hedging parameters in relation to the pricing methods is another important direction for future research, since Asian options are often used for hedging purposes.

Finally, it would be advantageous to find out if the upper bound mentioned in Chapter 8 is as good as the lower bound.
Appendix A

Monte Carlo simulation

A.1 Visual Basic code

The code can also be downloaded from the website http://www.lbn.dk/master/.

A.1.1 Standard Monte Carlo

Function GenerateRandomNumbers() As Variant

Dim pi As Double, a As Double, c As Double
Dim n As Long, n_max As Long
Dim I() As Double
Dim X() As Double

pi = 4 * Atn(1)
a = 7 ^ 5
c = 2 ^ 31 - 1
n_max = 6000000  '# random numbers

ReDim I(n_max)
ReDim X(n_max)

I(0) = 1773  '# seed for the linear congruential generator
For n = 1 To n_max  '# I() divided by c is iid U(0,1)
    I(n) = c * (a * I(n - 1) / c - Int(a * I(n - 1) / c))  '# a times I() modulo c
Next n

For n = 1 To n_max - 1 Step 2
'Box-Muller method: X() is iid N(0,1)
X(n) = Sqr(-2 * Log(I(n) / c)) * Cos(2 * pi * (I(n + 1) / c))
X(n + 1) = Sqr(-2 * Log(I(n) / c)) * Sin(2 * pi * (I(n + 1) / c))
Next n

Open ''c:\dokumenter\montecarlo\random_normal_seed_1773.txt'' _
For Output As #1
For n = 1 To n_max
Write #1, X(n)
Next n
Close #1

End Function

Function MonteCarlo(sigma As Double, K As Double, n_ave As Integer, _
q As Double, r As Double, S_0 As Double, T As Double) As Variant

Dim n As Long, n_path As Long, m As Long
Dim h As Double
Dim Z() As Double, S() As Double, AV() As Double
Dim asian() As Double, european() As Double
Dim V As Double, W As Double
Dim sdA As Double, sdE As Double
Dim output() As Double

'sigma: volatility
'K: strike price
'm_ave: # averaging times
'q: stock dividend
'r: interest rate
'S_0: stock price at time 0
'T: time to maturity

n_path = 50000  '# paths (replications) in the Monte Carlo simulation
h = T / m_ave  '# time between successive averaging times

ReDim Z(n_path, m_ave)
ReDim S(n_path, m_ave)
ReDim AV(n_path)
ReDim asian(n_path)
ReDim european(n_path)
A.1.  VISUAL BASIC CODE

ReDim output(3)

Open 'c:\dokumenter\montecarlo\random_normal_seed_1773.txt'

For Input As #2
For m = 1 To m_ave
    For n = 1 To n_path
        Input #2, Z(n, m)
    Next n
Next m
Close #2

For n = 1 To n_path  'stock price simulation
    S(n, 0) = S_0
    AV(n) = 0
    For m = 1 To m_ave
        S(n, m) = S(n, m - 1) * Exp((r - q - sigma ** 2 / 2) * h + _
            sigma * Sqr(h) * Z(n, m))  'simulated path
        AV(n) = AV(n) + S(n, m)
    Next m
    AV(n) = AV(n) / m_ave  'average stock price
Next n

For n = 1 To n_path  'option price simulation
    asian(n) = Exp(-r * T) * MaxFunc(AV(n) - K, 0)
    european(n) = Exp(-r * T) * MaxFunc(S(n, m_ave) - K, 0)
Next n

V = 0
W = 0

For n = 1 To n_path  'Monte Carlo simulation
    V = asian(n) + V
    W = european(n) + W
Next n

V = V / n_path  'Asian option price
W = W / n_path  'European option price

sdA = 0
sdE = 0
For n = 1 To n_path
    sdA = (asian(n) - V) ** 2 + sdA
    sdE = (european(n) - W) ** 2 + sdE
Next n
sda = Sqr(sda / (n_path - 1)) 'standard deviation of sample variance
sdE = Sqr(sdE / (n_path - 1))

output(0) = V
output(1) = sda / Sqr(n_path) 'standard deviation of estimator
output(2) = W
output(3) = sdE / Sqr(n_path)

MonteCarlo = output()

End Function

Function MaxFunc(number1 As Double, number2 As Double) As Double
If number1 > number2 Then
    MaxFunc = number1
ElseIf number1 <= number2 Then
    MaxFunc = number2
Else
    Stop
End If

End Function

Function BlackScholes(sigma, K, q, r, Su, T, u) As Double
Dim d1 As Double, d2 As Double

    d1 = (Log(Su / K) + (r - q + sigma ^2 / 2) * (T - u)) / (sigma * Sqr(T - u))
    d2 = d1 - sigma * Sqr(T - u)

BlackScholes = Su * WorksheetFunction.NormSDist(d1) - _
    Exp(-r * (T - u)) * K * WorksheetFunction.NormSDist(d2)

End Function

A.1.2 Antithetic method
Function MonteCarloAntithetic(sigma As Double, K As Double, _
A.1. VISUAL BASIC CODE

m_ave As Integer, q As Double, r As Double, S_0 As Double,
T As Double) As Variant

Dim n As Long, n_path As Long, m As Long
Dim h As Double
Dim Z() As Double, S() As Double, AV() As Double
Dim asian() As Double, european() As Double
Dim V As Double, W As Double
Dim sdA As Double, sdE As Double
Dim output() As Double

'sigma: volatility
'K: strike price
'm_ave: # averaging times
'q: stock dividend
'r: interest rate
'S_0: stock price at time 0
'T: time to maturity

n_path = 50000 ' # paths (replications) in the Monte Carlo simulation
h = T / m_ave 'time between successive averaging times

ReDim Z(n_path, m_ave)
ReDim S(2 * n_path, m_ave)
ReDim AV(2 * n_path)
ReDim asian(2 * n_path)
ReDim european(2 * n_path)
ReDim output(3)

Open ''c:\dokumenter\montecarlo\random_normal_seed_1773.txt'' _
For Input As #2
For m = 1 To m_ave
   For n = 1 To n_path
      Input #2, Z(n, m)
   Next n
Next m
Close #2

'stock price simulation
For n = 1 To n_path
   S(n, 0) = S_0
S(n + n_path, 0) = S_0
AV(n) = 0
AV(n + n_path) = 0
For m = 1 To m_ave
    S(n, m) = S(n, m - 1) * Exp((r - q - sigma^2 / 2) * h + _
        sigma * Sqr(h) * Z(n, m)) 'simulated path
    AV(n + n_path, m) = S(n + n_path, m - 1) * Exp((r - q - sigma^2 _
        / 2) * h + sigma * Sqr(h) * (-Z(n, m))) 'antithetic path
    AV(n) = AV(n) + S(n, m)
    AV(n + n_path) = AV(n + n_path) + S(n + n_path, m)
Next m
AV(n) = AV(n) / m_ave 'average stock price
AV(n + n_path) = AV(n + n_path) / m_ave
Next n

'option price simulation
For n = 1 To 2 * n_path
    asian(n) = Exp(-r * T) * MaxFunc(AV(n) - K, 0)
    european(n) = Exp(-r * T) * MaxFunc(S(n, m_ave) - K, 0)
Next n
V = 0
W = 0
For n = 1 To 2 * n_path 'Monte Carlo simulation
    V = asian(n) + V
    W = european(n) + W
Next n
V = V / (2 * n_path) 'Asian option price
W = W / (2 * n_path) 'European option price
sdA = 0
sdE = 0
For n = 1 To n_path
    sdA = ((asian(n) + asian(n + n_path)) / 2 - V)^2 + sdA
    sdE = ((european(n) + european(n + n_path)) / 2 - W)^2 + sdE
Next n
sdA = Sqr(sdA / (n_path - 1)) 'standard deviation _
     (square root of sample variance)
sdE = Sqr(sdE / (n_path - 1))
output(0) = V
A.1. VISUAL BASIC CODE

output(1) = sdA / Sqr(n_path) 'standard deviation of estimator
output(2) = W
output(3) = sdE / Sqr(n_path)

MonteCarloAntithetic = output()
End Function

A.1.3 Control variates

Function MonteCarloControl(sigma As Double, K As Double, _
    m_ave As Integer, q As Double, r As Double, S_0 As Double, _
    T As Double) As Variant

Dim n As Long, n_path As Long, m As Long
Dim h As Double
Dim Z() As Double, S() As Double, AV() As Double, AVgeo() As Double
Dim asian() As Double, european() As Double, asian_geo() As Double, _
    asian_new() As Double
Dim V As Double, W As Double, Vgeo As Double, Vnew As Double, _
    Vgeo_true As Double
Dim sdA As Double, sdE As Double, sdAgeo As Double, sdAnew As Double
Dim output() As Double

'sigma: volatility
'K: strike price
'm_ave: # averaging times
'q: stock dividend
'r: interest rate
'S_0: stock price at time 0
'T: time to maturity

n_path = 50000 ' # paths (replications) in the Monte Carlo simulation
h = T / m_ave ' time between successive averaging times

ReDim Z(n_path, m_ave)
ReDim S(n_path, m_ave)
ReDim AV(n_path)
ReDim AVgeo(n_path)
ReDim asian(n_path)
ReDim european(n_path)
ReDim asian_geo(n_path)
ReDim asian_new(n_path)
ReDim output(8)

Open "c:\dokumenter\montecarlo\random_normal_seed_1773.txt"
  For Input As #2
For m = 1 To m_ave
  For n = 1 To n_path
    Input #2, Z(n, m)
    Next n
  Next m
Close #2

'stock price simulation
For n = 1 To n_path
  S(n, 0) = S_0
  AV(n) = 0
  AVgeo(n) = 1
  For m = 1 To m_ave
    S(n, m) = S(n, m - 1) * Exp((-r - q - sigma^2 / 2) * h + _
      sigma * Sqr(h) * Z(n, m)) 'simulated path
    AV(n) = AV(n) + S(n, m)
    AVgeo(n) = AVgeo(n) * S(n, m)
  Next m
  AV(n) = AV(n) / m_ave 'average stock price
  AVgeo(n) = AVgeo(n) ^ (1 / m_ave) 'geometric average
  Next n

Vgeo_true = BS_geometric(sigma, K, m_ave, q, r, S_0, T) _
  'Analytical solution

'options price simulation
For n = 1 To n_path
  asian(n) = Exp(-r * T) * MaxFunc(AV(n) - K, 0)
  asian_geo(n) = Exp(-r * T) * MaxFunc(AVgeo(n) - K, 0)
  'europen(n) = Exp(-r * T) * MaxFunc(S(n, m_ave) - K, 0)
  asian_new(n) = asian(n) - asian_geo(n) + Vgeo_true
  Next n

'Monte Carlo simulation
V = 0
\begin{verbatim}
A.1. VISUAL BASIC CODE

Vgeo = 0
'W = 0
For n = 1 To n_path
    V = asian(n) + V
    Vgeo = asian_geo(n) + Vgeo
    'W = european(n) + W
Next n
V = V / n_path       'Asian option price
Vgeo = Vgeo / n_path  'Geometric Asian option price
'W = W / n_path       'European option price

Vnew = V - Vgeo + Vgeo_true
'Due control variate is the geometric option

'sdA = 0
sdAgeo = 0
'sdE = 0
sdAnew = 0
For n = 1 To n_path
    'sdA = (asian(n) - V)^2 + sdA
    sdAgeo = (asian_geo(n) - Vgeo)^2 + sdAgeo
    'sdE = (european(n) - W)^2 + sdE
    sdAnew = (asian_new(n) - Vnew)^2 + sdAnew
Next n
'sdA = Sqr(sdA / (n_path - 1))  'standard deviation
    (square root of sample variance)
sdAgeo = Sqr(sdAgeo / (n_path - 1))
'sdE = Sqr(sdE / (n_path - 1))
sdAnew = Sqr(sdAnew / (n_path - 1))

output(0) = Vnew
output(1) = sdAnew / Sqr(n_path)  'standard deviation of estimator
output(2) = Vgeo
output(3) = sdAgeo / Sqr(n_path)
output(4) = Vgeo_true
'output(5) = V
'output(6) = sdA / Sqr(n_path)
'output(7) = W
'output(8) = sdE / Sqr(n_path)

MonteCarloControl = output()
\end{verbatim}
End Function

Function BS_geometric(sigma, K, m_ave, q, r, S_0, T) As Double
Dim h As Double, mg As Double, sg As Double, d1 As Double,
    d2 As Double

    h = T / m_ave

    mg = Log(S_0) + (r - q - sigma ^ 2 / 2) * (T + h) / 2
    sg = Sqr(sigma ^ 2 * h * (2 * m_ave + 1) * (m_ave + 1) / (6 * m_ave))

    d1 = (mg - Log(K) + sg ^ 2) / sg
    d2 = d1 - sg

    BS_geometric = Exp(-r * T) * (Exp(mg + sg ^ 2 / 2) * _
        WorksheetFunction.NormSDist(d1) - _
        K * WorksheetFunction.NormSDist(d2))

End Function
Appendix B

Turnbull and Wakeman

B.1 Moments

To calculate the moments of the true distribution of the average, we use

\[ E[L_i^m] = E[(1 + R_i L_{i+1})^m], \quad i = 1, 2, ..., n, \quad m = 1, 2, ... \]  \hspace{1cm} (B.1)

For real numbers \( a \) and \( b \)

\[ (a + b)^2 = a^2 + b^2 + 2ab \]  \hspace{1cm} (B.2)
\[ (a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2 \]  \hspace{1cm} (B.3)
\[ (a + b)^4 = a^4 + b^4 + 4a^3b + 4ab^3 + 6a^2b^2 \]  \hspace{1cm} (B.4)

The first four moments of \( L_i \) are then

\[ E[L_i] = E[1 + R_i L_{i+1}] \]  \hspace{1cm} (B.5)
\[ E[L_i^2] = E[(1 + R_i L_{i+1})^2] \]
\[ = E[1 + R_i^2 L_{i+1}^2 + 2R_i L_{i+1}] \]
\[ = 1 + E[R_i^2]E[L_{i+1}^2] + 2E[R_i]E[L_{i+1}] \]  \hspace{1cm} (B.6)
\[ E[L_i^3] = E[(1 + R_i L_{i+1})^3] \]
\[ = E[1 + R_i^3 L_{i+1}^3 + 3R_i L_{i+1} + 3R_i^2 L_{i+1}^2] \]
\[ = 1 + E[R_i^3]E[L_{i+1}^3] + 3E[R_i^2]E[L_{i+1}^2] + 3E[R_i]E[L_{i+1}] \]  \hspace{1cm} (B.7)
\[ E[L_i^4] = E[(1 + R_i L_{i+1})^4] \]
\[ = E[1 + R_i^4 L_{i+1}^4 + 4R_i L_{i+1} + 4R_i^3 L_{i+1} + 6R_i^2 L_{i+1}^2] \]
\[ + 4E[R_i]E[L_{i+1}] \]  \hspace{1cm} (B.8)
B.2 Cumulants

For real numbers $a$ and $b$

\[
\begin{align*}
(a - b)^2 &= a^2 + b^2 - 2ab \\
(a - b)^3 &= a^3 - b^3 - 3a^2b + 3ab^2 \\
(a - b)^4 &= a^4 + b^4 - 4a^3b - 4ab^3 + 6a^2b^2
\end{align*}
\]

Let $Y$ have distribution $F$. Then the cumulants are

\[
\begin{align*}
\kappa_1(F) &= E[Y] \\
\kappa_2(F) &= E[(Y - E[Y])^2] \\
&= E[Y^2] - E[Y]^2 \quad \text{(B.9)} \\
\kappa_3(F) &= E[(Y - E[Y])^3] \\
&= E[Y^3] + 3E[Y^2]E[Y] \quad \text{(B.10)} \\
\kappa_4(F) &= E[(Y - E[Y])^4] - 3(\kappa_2(F))^2 \\
&\quad - 3(E[Y^2] - E[Y]^2)^2 \\
&\quad - 3E[Y^2]^2 \quad \text{(B.11)}
\end{align*}
\]

[Turnbull and Wakeman, 1991] and [Jarrow and Rudd, 1982] are inconsistent regarding the definition of $\kappa_4$. The first two authors subtract $3\kappa_2(F)$ instead of $3(\kappa_2(F))^2$, which is probably due to a typographical error. Numerically, this doesn’t make any difference in our setup, because we have set the first two moments equal in the two distributions.

B.3 Moment matching

Matching the moments of the approximating lognormal distribution to the true distribution requires us to solve the following system of equations
\[ \alpha_1(F) = \alpha_1(G) \]
\[ \alpha_2(F) = \alpha_2(G) \]
\[ \ln \alpha_1(F) = \nu + \frac{1}{2} \lambda^2 \]
\[ \ln \alpha_2(F) = 2(\ln \alpha_1(F) - \frac{1}{2} \lambda^2) + 2\lambda^2 \]
\[ \lambda^2 = \ln \alpha_2(F) - 2 \ln \alpha_1(F) \]
\[ \nu = 2 \ln \alpha_1(F) - \frac{1}{2} \ln \alpha_2(F) \]

\[ \lambda^2 = \ln \alpha_2(F) - 2 \ln \alpha_1(F) \]

\section*{B.4 Lognormal density and derivatives}

The density function \( g(x) \) for the lognormal distribution with parameters \( \nu \) and \( \lambda^2 \) is given by

\[ g(x) = \frac{1}{\lambda \sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{(\nu - \ln x)^2}{2\lambda^2}\right), \quad x > 0 \quad (B.13) \]

The first derivative of \( g \) with respect to \( x \) is

\[
\frac{dg}{dx}(x) = \frac{1}{\lambda \sqrt{2\pi}} \frac{-1}{x^2} \exp\left(-\frac{(\nu - \ln x)^2}{2\lambda^2}\right) + \frac{1}{\lambda \sqrt{2\pi}} \frac{1}{x} \exp\left(-\frac{(\nu - \ln x)^2}{2\lambda^2}\right) \frac{-1}{2\lambda^2}(\nu - \ln x) \frac{-1}{x} \\
= g(x)(\frac{-1}{x} + \frac{1}{x} \frac{1}{\lambda^2}(\nu - \ln x)) \\
= g(x)\frac{1}{x}(\frac{\nu - \ln x}{\lambda^2} - 1), \quad x > 0 \quad (B.14) \]
APPENDIX B. TURNBULL AND WAKEMAN

Set
\[
\phi(x) = \frac{\nu - \ln x}{\lambda^2} - 1, \quad x > 0
\] (B.15)

The second derivative is then
\[
\frac{d^2 g}{dx^2}(x) = (g(x)\frac{1}{x}\phi(x))' = g'(x)\frac{1}{x}\phi(x) + g(x)(\frac{-1}{x^2}\phi(x) + \frac{1}{x}\phi'(x)) \\
g(x)\frac{1}{x^2}(\phi(x))^2 - g(x)\frac{1}{x^2}\phi(x) + g(x)\frac{1}{x^2} - \frac{1}{x^2} \\
= g(x)\frac{1}{x^2}(\phi(x)(\phi(x) - 1) - \frac{1}{\lambda^2}) \\
= g(x)\frac{1}{x^2}(\frac{\nu - \ln x}{\lambda^2} - 1)(\frac{\nu - \ln x}{\lambda^2} - 2) - \frac{1}{\lambda^2}, \quad x > 0 \] (B.16)

B.5 Visual Basic code

Function TurnbullWakeman(sigma As Double, K As Double, _
 n As Integer, q As Double, r As Double, S_0 As Double, _
 T As Double) As Variant
Dim sigSq As Double, h As Double, k2 As Double
Dim R1 As Double, R2 As Double, R3 As Double, R4 As Double
Dim L1 As Double, L2 As Double, L3 As Double, L4 As Double
Dim j As Integer
Dim Y1 As Double, Y2 As Double, Y3 As Double, Y4 As Double
Dim A1 As Double, A2 As Double, A3 As Double, A4 As Double
Dim my_approx As Double, sigSq_approx As Double
Dim c2 As Double, c3 As Double, c4 As Double
Dim d1 As Double, d2 As Double
Dim price As Double
Dim term1 As Double, term2 As Double, term3 As Double
Dim Price2 As Double, Price3 As Double, PriceAllTerms As Double
Dim output(8) As Double

' sigma: volatility
' K: strike price
' n: # averaging times
' q: stock dividend
' r: interest rate
B.5. VISUAL BASIC CODE

'S_0: stock price at time 0
'T: time to maturity

sigSq = sigma ^2
h = T / n  'time increment between averaging dates
k2 = K * n / S_0  'scaled strike price - (lower case) k in TW article

'PART 1 - calculate moments of true distribution
'i=T-(n-1) in TW article
R1 = LogMoment(1, (r - q - sigSq / 2) * h, sigSq * h)
R2 = LogMoment(2, (r - q - sigSq / 2) * h, sigSq * h)
R3 = LogMoment(3, (r - q - sigSq / 2) * h, sigSq * h)
R4 = LogMoment(4, (r - q - sigSq / 2) * h, sigSq * h)
L1 = 1  'L(T+1)=L(i+n)=1
L2 = 1
L3 = 1
L4 = 1

For j = (n - 1) To 1 Step -1  'time i+j, i.e. time T to time T-n+2
  'Priority L4,L3,L2,L1 important!
  L4 = 1 + R4 * L4 + 4 * R3 * L3 + 6 * R2 * L2 + 4 * R1 * L1
  L3 = 1 + R3 * L3 + 3 * R2 * L2 + 3 * R1 * L1
  L2 = 1 + R2 * L2 + 2 * R1 * L1
  L1 = 1 + R1 * L1
Next j

'Moments of true distribution
Y1 = R1 * L1
Y2 = R2 * L2
Y3 = R3 * L3
Y4 = R4 * L4

'PART 2 - calculate moments of approximating distribution
'match moments of approx. dist. to true dist.
my_approx = 2 * Log(Y1) - Log(Y2) / 2
sigSq_approx = Log(Y2) - 2 * Log(Y1)

'Moments of approximating distribution
A1 = Y1  'per construction
A2 = Y2  'per construction
A3 = LogMoment(3, my_approx, sigSq_approx)
\( A4 = \text{LogMoment}(4, \text{my\_approx}, \text{sigSq\_approx}) \)

'PART 3 - calculate option price
\( c2 = Y2 - Y1 \cdot ^2 - (A2 - A1 \cdot ^2) \)
\( c3 = Y3 + 2 \cdot Y1 \cdot ^3 - 3 \cdot Y2 \cdot Y1 \)
\( c3 = c3 - (A3 + 2 \cdot A1 \cdot ^3 - 3 \cdot A2 \cdot A1) \)
\( c4 = Y4 - 6 \cdot Y1 \cdot ^4 - 4 \cdot Y3 \cdot Y1 + 12 \cdot Y2 \cdot Y1 \cdot ^2 - 3 \cdot Y2 \cdot ^2 \)
\( c4 = c4 - (A4 - 6 \cdot A1 \cdot ^4 - 4 \cdot A3 \cdot A1 + 12 \cdot A2 \cdot A1 \cdot ^2 - 3 \cdot A2 \cdot ^2) \)
\( c4 = c4 + 3 \cdot c2 \cdot ^2 \)
\( d1 = (-\text{Log}(k2) + \text{my\_approx} + \text{sigSq\_approx}) / \text{Sqr}(\text{sigSq\_approx}) \)
\( d2 = d1 - \text{Sqr}(\text{sigSq\_approx}) \)
\( \text{price} = \text{Exp}(\text{my\_approx} + \text{sigSq\_approx} / 2) \cdot _{-} \)
\( \text{WorksheetFunction.} \text{NormSDist}(d1) - _{-} \)
\( k2 * \text{WorksheetFunction.} \text{NormSDist}(d2) \)
\( ' \text{price} = \text{price} + c2 / 2 * \text{LogDensity}(0, k2, \text{my\_approx}, \text{sigSq\_approx}) \)
\( ' \text{price} = \text{price} - c3 / 6 * \text{LogDensity}(1, k2, \text{my\_approx}, \text{sigSq\_approx}) \)
\( ' \text{price} = \text{price} + c4 / 24 * \text{LogDensity}(2, k2, \text{my\_approx}, \text{sigSq\_approx}) \)
\( \text{price} = \text{price} * \text{Exp}(-r * T) * S_0 / n \)

'term1=0 in this setup since c2=0
\( \text{term1} = c2 / 2 * \text{LogDensity}(0, k2, \text{my\_approx}, \text{sigSq\_approx}) \cdot _{-} \)
\( \text{Exp}(-r * T) * S_0 / n \)
\( \text{term2} = -c3 / 6 * \text{LogDensity}(1, k2, \text{my\_approx}, \text{sigSq\_approx}) \cdot _{-} \)
\( \text{Exp}(-r * T) * S_0 / n \)
\( \text{term3} = c4 / 24 * \text{LogDensity}(2, k2, \text{my\_approx}, \text{sigSq\_approx}) \cdot _{-} \)
\( \text{Exp}(-r * T) * S_0 / n \)

Price2 = price + term1
Price3 = price + term1 + term2
PriceAllTerms = price + term1 + term2 + term3

output(0) = PriceAllTerms 'TW price
output(1) = Price3 'No correction for diff. in 4. cumulants
output(2) = Price2 'Levy price, no corr. for diff. in cumulants
output(3) = term2 'correction for diff. in 3. cumulants
output(4) = term3 'correction for diff. in 4. cumulants
output(5) = c3 / 6
output(6) = LogDensity(1, k2, my\_approx, sigSq\_approx)
output(7) = c4 / 24
output(8) = LogDensity(2, k2, my_approx, sigSq_approx)

TurnbullWakeman = output()

End Function

Function LogMoment(m As Integer, my As Double, sigSq As Double) As Double
    'Returns the m'th moment of logN(my, sigSq)
    LogMoment = Exp(my * m + sigSq * m ^ 2 / 2)
End Function

Function LogDensity(deriv As Integer, x As Double, my As Double, sigSq As Double) As Double
    'Density function (and derivatives) of logN(my, sigSq)
    Dim pi As Double, frac As Double, f As Double
    pi = 4 * Atn(1)
    frac = (my - Log(x)) / sigSq
    f = 1 / (Sqr(2 * pi * sigSq) * x) * Exp(-(my - Log(x)) ^ 2 / (2 * sigSq))
    If deriv = 0 Then
        'Lognormal density f(x)
        LogDensity = f
    ElseIf deriv = 1 Then
        '1. order derivative f'(x)
        LogDensity = f / x * (frac - 1)
    ElseIf deriv = 2 Then
        '2. order derivative f''(x)
        LogDensity = f / x ^ 2 * ((frac - 1) * (frac - 2) - 1 / sigSq)
    Else
        Stop
    End If
End Function
Appendix C

Milevsky and Posner

C.1 Gamma distribution

Proposition 9 states that

\[ G(x|\alpha, \beta) = G(\frac{x}{\beta} | \alpha, 1), \quad \forall x > 0 \]  
\[ g(x|\alpha, \beta) = \frac{1}{\beta} g(\frac{x}{\beta} | \alpha, 1), \quad \forall x > 0 \]  
\[ g(x|\alpha, \beta) = \frac{x}{\beta(\alpha - 1)} g(x|\alpha - 1, \beta), \quad \forall x > 0, \quad \forall \alpha > 1 \]  

Proof of Proposition 9. Using the definition of the density function

\[ G(x|\alpha, \beta) = \int_0^\infty g(u|\alpha, \beta)du = \int_0^\infty \Gamma(\alpha)^{-1} \beta^{-\alpha} u^{\alpha-1} \exp(-\beta^{-1}u)du \]  
\[ = \int_0^\infty \Gamma(\alpha)^{-1} \beta^{-\alpha} \beta^{-\alpha-1} v^{\alpha-1} \exp(-v)\beta dv \]  
\[ = \int_0^\infty \Gamma(\alpha)^{-1} v^{\alpha-1} \exp(-v)dv \]  
\[ = G(\frac{x}{\beta} | \alpha, 1), \quad \forall x > 0 \]

where we have made the change of variables \( v = \frac{u}{\beta} \) so \( \frac{dv}{du} = \frac{1}{\beta} \). This proves (C.1). (C.2) follows from the above since

\[ g(x|\alpha, \beta) = \frac{d}{dx}(G(x|\alpha, \beta)) = \frac{d}{dx}(G(\frac{x}{\beta} | \alpha, 1)) \]  
\[ = \frac{1}{\beta} g(\frac{x}{\beta} | \alpha, 1), \quad \forall x > 0 \]
For $\alpha > 1$

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

which we use to prove (C.3)

$$g(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha - 1)\beta^{\alpha}} x^{\alpha-1} \exp(-\beta^{-1}x)$$

$$= \frac{1}{(\alpha - 1)\Gamma(\alpha - 1)\beta^{\alpha}} x^{\alpha} \exp(-\beta^{-1}x)$$

$$= \frac{x}{\beta(\alpha - 1)} g(x|\alpha - 1, \beta), \quad \forall x > 0, \quad \forall \alpha > 1$$

$$\blacksquare$$

C.2  Reciprocal gamma distribution

C.2.1  Moments

Let $X \sim \Gamma(\alpha, \beta)$ and $Y = \frac{1}{X}$. $Y$ is reciprocally gamma distributed by definition and its moments of order $m < \alpha$ are

$$E[Y^m] = E[\frac{1}{X^m}] = \int_0^{\infty} \frac{1}{x^m} g(x|\alpha, \beta)dx$$

$$= \int_0^{\infty} \frac{1}{x^{m-1} \beta(\alpha - 1)} g(x|\alpha - 1, \beta)dx$$

$$= \int_0^{\infty} \frac{1}{x^{m-2} \beta^2(\alpha - 1)(\alpha - 2)} g(x|\alpha - 2, \beta)dx$$

$$= \ldots$$

$$= \int_0^{\infty} \frac{1}{\beta^m(\alpha - 1)(\alpha - 2)\ldots(\alpha - m)} g(x|\alpha - m, \beta)dx$$

$$= \frac{1}{\beta^m(\alpha - 1)(\alpha - 2)\ldots(\alpha - m)} \int_0^{\infty} g(x|\alpha - m, \beta)dx$$

$$= \frac{1}{\beta^m(\alpha - 1)(\alpha - 2)\ldots(\alpha - m)}$$

(C.4)

where we have used (C.3) successively.

The first two (positive) moments of the reciprocal gamma distribution are, cf. (C.4)

$$M_1 = \frac{1}{\beta(\alpha - 1)}$$

(C.5)
and
\[ M_2 = \frac{1}{\beta^2(\alpha - 1)(\alpha - 2)} \] (C.6)

We want to solve for \( \alpha \) and \( \beta \) in these equations. (C.5) and (C.6) can be written as
\[ \beta = \frac{1}{M_1(\alpha - 1)} \] (C.7)

and
\[ \beta^2 = \frac{1}{M_2(\alpha - 1)(\alpha - 2)} \] (C.8)

Inserting (C.7) into (C.8) gives
\[
\frac{1}{M_1(\alpha - 1)^2} = \frac{1}{M_2(\alpha - 1)(\alpha - 2)}
\]
\[
\Rightarrow \quad M_1^2(\alpha - 1) = M_2(\alpha - 1) - M_2
\]
\[
\Rightarrow \quad (M_1^2 - M_2)(\alpha - 1) = -M_2
\]
\[
\Rightarrow \quad (\alpha - 1) = \frac{-M_2}{M_1^2 - M_2}
\]
\[
\Rightarrow \quad \alpha = \frac{2M_2 - M_1^2}{M_1^2 - M_2}
\]

Inserting the last line but one into (C.7) gives
\[
\beta = (M_1 \frac{M_2}{M_2 - M_1^2})^{-1} = \frac{M_2 - M_1^2}{M_1 M_2} \] (C.9)

All in all we have the parameters \( \alpha \) and \( \beta \) expressed in terms of the first two moments
\[
\alpha = \frac{2M_2 - M_1^2}{M_1^2 - M_2} \quad \text{(C.10)}
\]
\[
\beta = \frac{M_2 - M_1^2}{M_1 M_2} \quad \text{(C.11)}
\]

### C.2.2 Density and derivatives

The density function \( g_R \) of the reciprocal gamma distribution is
\[
g_R(y) = \frac{1}{y^2} g\left(\frac{1}{y}\right)
\]
\[
= \frac{1}{y^2} \Gamma(\alpha)^{-1} \beta^{-\alpha} \left(\frac{1}{y}\right)^{\alpha-1} \exp(-\beta^{-1} \frac{1}{y})
\]
\[
= \Gamma(\alpha)^{-1} \beta^{-\alpha} y^{-\alpha-1} \exp(-\beta^{-1} y^{-1}), \quad y > 0 \] (C.12)
where \( g \) is the density function of the gamma distribution, cf. (6.1). Set

\[
\begin{align*}
  c(\alpha, \beta) & = \Gamma(\alpha)^{-1} \beta^{-\alpha} \\
  h(y) & = y^{-\alpha-1} \exp(-\beta^{-1} y^{-1}), \quad y > 0
\end{align*}
\]

so

\[ g_R(y) = c(\alpha, \beta) h(y) \]

The first derivative of \( h \) with respect to \( y \) is

\[
\frac{dh}{dy}(y) = y^{-\alpha-1} \exp(-\beta^{-1} y^{-1}) \beta^{-1} y^{-2} + (-\alpha - 1) y^{-\alpha-2} \exp(-\beta^{-1} y^{-1})
\]

\[ = \exp(-\beta^{-1} y^{-1}) (\beta^{-1} y^{-\alpha-3} - (\alpha + 1) y^{-\alpha-2}) \]

\[ = h(y) (\beta^{-1} y^{-2} - (\alpha + 1) y^{-1}) \]

The second derivative is then

\[
\frac{d^2h}{dy^2}(y) = h'(y) (\beta^{-1} y^{-2} - (\alpha + 1) y^{-1})
\]

\[ + h(y) (-2 \beta^{-1} y^{-3} + (\alpha + 1) y^{-2}) \]

\[ = h(y) (\beta^{-1} y^{-2} - (\alpha + 1) y^{-1})^2 \]

\[ + h(y) (-2 \beta^{-1} y^{-3} + (\alpha + 1) y^{-2}) \]

\[ = h(y) (\beta^{-2} y^{-4} + (\alpha + 1)^2 y^{-2} - 2 \beta^{-1} y^{-2} (\alpha + 1) y^{-1}) \]

\[ + h(y) (-2 \beta^{-1} y^{-3} + (\alpha + 1) y^{-2}) \]

\[ = h(y) (\beta^{-2} y^{-4} - 2 \beta^{-1} (\alpha + 2) y^{-3} + (\alpha^2 + 3 \alpha + 2) y^{-2}) \]

From the above, the first and second order derivatives of \( g_R \), for \( y > 0 \), are

\[
\frac{dg_R}{dy}(y) = c(\alpha, \beta) h'(y)
\]

\[ = g_R(y) (\beta^{-1} y^{-2} - (\alpha + 1) y^{-1}) \]

\[ = \Gamma(\alpha)^{-1} (\beta^{-\alpha-1} y^{-\alpha-3} - (\alpha + 1) \beta^{-\alpha} y^{-\alpha-2}) \]

\[ \cdot \exp(-\beta^{-1} y^{-1}) \]

and

\[
\frac{d^2g_R}{dy^2}(y) = c(\alpha, \beta) h''(y)
\]

\[ = g_R(y) (\beta^{-2} y^{-4} - 2 \beta^{-1} (\alpha + 2) y^{-3} + (\alpha^2 + 3 \alpha + 2) y^{-2}) \]

\[ = \Gamma(\alpha)^{-1} \{ \beta^{-\alpha-2} y^{-\alpha-5} - 2 \beta^{-\alpha-1} (\alpha + 2) y^{-\alpha-4} \]

\[ + (\alpha^2 + 3 \alpha + 2) \beta^{-\alpha} y^{-\alpha-3} \} \exp(-\beta^{-1} y^{-1}) \]
C.3 Moments

Formulae for the first two moments of the arithmetic average are given in [Levy, 1992]. His average is

$$A_{\text{Levy}} = \frac{1}{n+1} \sum_{i=0}^{n} S_{ti}$$  \hspace{1cm} (C.20)

so we have to modify them to our specification of the average

$$A = \frac{1}{n} \sum_{i=1}^{n} S_{ti}$$  \hspace{1cm} (C.21)

This is done as follows

$$A_{\text{Levy}} = \frac{1}{n+1} \sum_{i=0}^{n} S_{ti} = \frac{n}{n(n+1)} (S_0 + \sum_{i=1}^{n} S_{ti})$$

$$= \frac{S_0}{n+1} + \frac{n}{n+1} \cdot \frac{1}{n} \sum_{i=1}^{n} S_{ti} = \frac{S_0}{n+1} + \frac{n}{n+1} A$$  \hspace{1cm} (C.22)

Using (C.22), the first moment is

$$E[A_{\text{Levy}}] = E\left[ \frac{S_0}{n+1} + \frac{n}{n+1} A \right]$$

$$= \frac{S_0}{n+1} + \frac{n}{n+1} E[A]$$  \hspace{1cm} (C.23)

and the second moment is

$$E[A_{\text{Levy}}^2] = E\left[\left( \frac{S_0}{n+1} + \frac{n}{n+1} A \right)^2 \right]$$

$$= E\left[\left( \frac{S_0}{n+1} \right)^2 + \left( \frac{n}{n+1} A \right)^2 + 2 \frac{S_0}{n+1} \frac{n}{n+1} A \right]$$

$$= \frac{n^2}{(n+1)^2} E[A^2] + 2 S_0 \frac{n}{(n+1)^2} E[A] + \frac{S_0^2}{(n+1)^2}$$  \hspace{1cm} (C.24)

Inverting (C.23) and (C.24) gives

$$E[A] = \frac{1}{n} ((n+1)E[A_{\text{Levy}}] - S_0)$$  \hspace{1cm} (C.25)

and

$$E[A^2] = \frac{1}{n^2} ((n+1)^2 E[A_{\text{Levy}}^2] - 2 S_0 n E[A] - S_0^2)$$

$$= \frac{1}{n^2} ((n+1)^2 E[A_{\text{Levy}}^2] - 2 S_0 (n+1) E[A_{\text{Levy}}] + S_0^2)$$  \hspace{1cm} (C.26)
Levy’s formulae with $\xi = 0, \ m = 0, \ t = 0, \ T_0 = 0$ (in his notation) are

$$E[A_{Levy}] = \frac{S_0}{n+1} + \frac{S_0}{n+1} e^{(r-q)h} \frac{1 - e^{(r-q)n h}}{1 - e^{(r-q)h}} \quad (C.27)$$

and

$$E[A_{Levy}^2] = \left(\frac{S_0}{n+1}\right)^2 + 2\left(\frac{S_0}{n+1}\right)^2 e^{(r-q)h} \frac{1 - e^{(r-q)n h}}{1 - e^{(r-q)h}}$$

\[\left(\frac{S_0}{n+1}\right)^2 (a_1 - a_2 + a_3 - a_4) \]

\[= \frac{S_0}{n+1} (2E[A_{Levy}] - \frac{S_0}{n+1}) + \left(\frac{S_0}{n+1}\right)^2 (a_1 - a_2 + a_3 - a_4) \]

\[= \frac{S_0}{n+1} (2E[A_{Levy}] + \frac{S_0}{n+1} (a_1 - a_2 + a_3 - a_4 - 1)) \quad (C.28)\]

where

\[a_1 = \frac{e^{2(r-q)+\sigma^2}h - e^{2(r-q)+\sigma^2}(n+1)h}{(1 - e^{(r-q)h})(1 - e^{(2(r-q)+\sigma^2)h}))} \quad (C.29)\]

\[a_2 = \frac{e^{(r-q)(n+2)+\sigma^2}h - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h}))} \quad (C.30)\]

\[a_3 = \frac{e^{3(r-q)+\sigma^2}h - e^{(r-q)(n+2)+\sigma^2}h}{(1 - e^{(r-q)h})(1 - e^{(r-q+\sigma^2)h}))} \quad (C.31)\]

\[a_4 = \frac{e^{4(r-q)+2\sigma^2}h - e^{(2(r-q)+\sigma^2)(n+1)h}}{(1 - e^{(r-q+\sigma^2)h})(1 - e^{(2(r-q)+\sigma^2)h}))} \quad (C.32)\]

The modified expressions for the moments are now obtained from (C.25) and (C.26)

$$E[A] = \frac{S_0}{n} e^{(r-q)h} \frac{1 - e^{(r-q)n h}}{1 - e^{(r-q)h}} \quad (C.33)$$

and

$$E[A^2] = \frac{S_0^2}{n^2} (a_1 - a_2 + a_3 - a_4) \quad (C.34)$$

where the $a_i$ are given by (C.29)-(C.32).
C.4 Visual Basic code

Function MilevskyPosner(sigma As Double, K As Double, n As Integer, _
    q As Double, r As Double, S_0 As Double, T As Double) As Double
Dim sigSq As Double, h As Double, g As Double
Dim A1 As Double, A2 As Double, A3 As Double, A4 As Double
Dim M1 As Double, M2 As Double
Dim alpha As Double, beta As Double
Dim G1 As Double, G2 As Double
Dim price As Double

' sigma: volatility
' K: strike price
' n: # averaging times
' q: stock dividend
' r: interest rate
' S_0: stock price at time 0
' T: time to maturity

sigSq = sigma ^ 2
h = T / n   ' time increment between averaging dates
g = r - q

' adjusted Levy formula for 1. and 2. moment of average
If g = 0 Then
    M1 = S_0
    ' continuous time M2
    M2 = 2 * S_0 ^ 2 / T ^ 2 * (Exp(sigSq * T) - 1 - sigSq * T) / sigSq ^ 2
    Stop   ' r = q (g = 0) not implemented in discrete time!
Else
    M1 = S_0 / n * Exp(g * h) * (1 - Exp(g * n * h)) / (1 - Exp(g * h))

    A1 = (1 - Exp(g * h)) * (1 - Exp((2 * g + sigSq) * h))
    A1 = (Exp((2 * g + sigSq) * h) - Exp((2 * g + sigSq) * _
            (n + 1) * h)) / A1
    A2 = (1 - Exp(g * h)) * (1 - Exp((g + sigSq) * h))
    A2 = (Exp((g * (n + 2) + sigSq) * h) - Exp((2 * g + sigSq) * _
            (n + 1) * h)) / A2
    A3 = (1 - Exp(g * h)) * (1 - Exp((g + sigSq) * h))
    A3 = (Exp((3 * g + sigSq) * h) - Exp((g * (n + 2) + sigSq) * h)) / A3
    A4 = (1 - Exp((g + sigSq) * h)) * (1 - Exp((2 * g + sigSq) * h))
\[ A_4 = \frac{\text{Exp}(2 \cdot g + \text{sigSq} \cdot 2 \cdot h) - \text{Exp}(2 \cdot g + \text{sigSq} \cdot (n + 1) \cdot h)}{A_4} \]

\[ M_2 = S_0^2 \cdot n^2 \cdot (A_1 - A_2 + A_3 - A_4) \]

End If

'Moment matching
alpha = (2 \cdot M_2 - M_1^2) / (M_2 - M_1^2)
beta = (M_2 - M_1^2) / (M_2 \cdot M_1)

G1 = WorksheetFunction.GammaDist(1 / K, alpha - 1, beta, True)
G2 = WorksheetFunction.GammaDist(1 / K, alpha, beta, True)

price = \text{Exp}(-r \cdot T) \cdot (M_1 \cdot G_1 - K \cdot G_2) \quad \text{'Asian option price}

MilevskyPosner = price

End Function

Function MilevskyPosnerContinuous(sigma As Double, K As Double, _
    n As Integer, q As Double, r As Double, S_0 As Double, _
    T As Double) As Double
    Dim sigSq As Double
    Dim M1 As Double, M2 As Double, M2_sub As Double
    Dim alpha As Double, beta As Double
    Dim G1 As Double, G2 As Double
    Dim price As Double

    sigSq = sigma^2
    'The variable n is not used!

    If r = q Then
        M1 = S_0
        M2 = 2 \cdot S_0^2 \cdot T^2 \cdot (\text{Exp}(\text{sigSq} \cdot T) - 1 - \text{sigSq} \cdot T) / \text{sigSq}^2
    Else
        M1 = S_0 \cdot (\text{Exp}((r - q) \cdot T) - 1)
        M1 = M1 / ((r - q) \cdot T)

        M2_sub = (r - q + \text{sigSq}) \cdot (2 \cdot r - 2 \cdot q + \text{sigSq})
        M2_sub = \text{Exp}(2 \cdot (r - q) + \text{sigSq} \cdot T) / M2_sub
    End If

    MilevskyPosnerContinuous = price

End Function
C.4. VISUAL BASIC CODE

\[
\begin{align*}
M2 &= 1 / (2 * (r - q) + \text{sigSq}) \\
M2 &= M2 - \text{Exp}((r - q) * T) / (r - q + \text{sigSq}) \\
M2 &= M2 / (r - q) + M2_{sub} \\
M2 &= M2 * 2 * S_0 ^ 2 / T ^ 2 \\
\end{align*}
\]

End If

'Momentmatching
alpha = (2 * M2 - M1 ^ 2) / (M2 - M1 ^ 2)
beta = (M2 - M1 ^ 2) / (M2 * M1)

G1 = WorksheetFunction.GammaDist(1 / K, alpha - 1, beta, True)
G2 = WorksheetFunction.GammaDist(1 / K, alpha, beta, True)

price = Exp(-r * T) * (M1 * G1 - K * G2)  'Asian option price

MilevskyPosnerContinuous = price

End Function

Function MPimproved(sigma As Double, K As Double, n As Integer, _
q As Double, r As Double, S_0 As Double, T As Double) As Double

'Uses the TW method in a MP setup
Dim sigSq As Double, h As Double, k2 As Double
Dim R1 As Double, R2 As Double, R3 As Double, R4 As Double
Dim L1 As Double, L2 As Double, L3 As Double, L4 As Double
Dim j As Integer
Dim Y1 As Double, Y2 As Double, Y3 As Double, Y4 As Double
Dim A1 As Double, A2 As Double, A3 As Double, A4 As Double
Dim alfa As Double, beta As Double
Dim c2 As Double, c3 As Double, c4 As Double
Dim G1 As Double, G2 As Double
Dim price As Double

'sigma: volatility
'K:    strike price
'n:    # averaging times
'q:    stock dividend
'r:    interest rate
'S_0:  stock price at time 0
'T:    time to maturity
sigSq = sigma ^ 2
h = T / n  'time increment between averaging dates
k2 = n * K / S_0  '(lower case) k in TW article

'PART 1 - calculate moments of true distribution
'i=T-(n-1) in TW article
R1 = LogMoment(1, (r - q - sigSq / 2) * h, sigSq * h)
R2 = LogMoment(2, (r - q - sigSq / 2) * h, sigSq * h)
R3 = LogMoment(3, (r - q - sigSq / 2) * h, sigSq * h)
R4 = LogMoment(4, (r - q - sigSq / 2) * h, sigSq * h)
L1 = 1  'L(T+1)=L(i+n)=1
L2 = 1
L3 = 1
L4 = 1

For j = (n - 1) To 1 Step -1  'time i+j, i.e. time T to time T-n+2
    'Priority L4,L3,L2,L1 important!
    L4 = 1 + R4 * L4 + 4 * R3 * L3 + 6 * R2 * L2 + 4 * R1 * L1
    L3 = 1 + R3 * L3 + 3 * R2 * L2 + 3 * R1 * L1
    L2 = 1 + R2 * L2 + 2 * R1 * L1
    L1 = 1 + R1 * L1
Next j

'Moments of true distribution
Y1 = R1 * L1
Y2 = R2 * L2
Y3 = R3 * L3
Y4 = R4 * L4

'PART 2 - calculate moments of approximating distribution
'match moments of approx. dist. to true dist.
alfa = (2 * Y2 - Y1 ^ 2) / (Y2 - Y1 ^ 2)
beta = (Y2 - Y1 ^ 2) / (Y1 * Y2)

'Moments of approximating distribution
A1 = Y1  'per construction
A2 = Y2  'per construction
A3 = RGMoment(3, alfa, beta)
A4 = RGMoment(4, alfa, beta)

'PART 3 - calculate option price
C.4. VISUAL BASIC CODE

\[\begin{align*}
c2 &= Y2 - Y1 \cdot 2 - (A2 - A1 \cdot 2) \\
c3 &= Y3 + 2 \cdot Y1 \cdot 3 - 3 \cdot Y2 \cdot Y1 \\
c3 &= c3 - (A3 + 2 \cdot A1 \cdot 3 - 3 \cdot A2 \cdot A1) \\
c4 &= Y4 - 6 \cdot Y1 \cdot 4 - 4 \cdot Y3 \cdot Y1 + 12 \cdot Y2 \cdot Y1 \cdot 2 - 3 \cdot Y2 \cdot 2 \\
c4 &= c4 - (A4 - 6 \cdot A1 \cdot 4 - 4 \cdot A3 \cdot A1 + 12 \cdot A2 \cdot A1 \cdot 2 - 3 \cdot A2 \cdot 2) \\
c4 &= c4 + 3 \cdot c2 \cdot 2
\end{align*}\]

\[\begin{align*}
G1 &= \text{WorksheetFunction.GammaDist}(1 / k2, \text{alfa} - 1, \text{beta}, \text{True}) \\
G2 &= \text{WorksheetFunction.GammaDist}(1 / k2, \text{alfa}, \text{beta}, \text{True})
\end{align*}\]

\[\begin{align*}
\text{price} &= Y1 \cdot G1 - k2 \cdot G2 \\
\text{price} &= \text{price} + c2 / 2 \cdot \text{RGDensity}(0, k2, \text{alfa}, \text{beta}) \\
\text{price} &= \text{price} - c3 / 6 \cdot \text{RGDensity}(1, k2, \text{alfa}, \text{beta}) \\
\text{price} &= \text{price} + c4 / 24 \cdot \text{RGDensity}(2, k2, \text{alfa}, \text{beta}) \\
\text{price} &= \text{price} \cdot \text{Exp}(-r \cdot T) \cdot S_0 / n
\end{align*}\]

MPimproved = price

End Function

Function RGMoment(m As Integer, \text{alfa} As Double, _
betAs Double) As Double
'Returns the m\text{'th moment of reciprocal gamma(alfa, beta)

If \text{alfa} > m Then
  If m = 1 Then
    RGMoment = 1 / (beta \cdot (\text{alfa} - 1))
  ElseIf m = 2 Then
    RGMoment = 1 / (beta \cdot 2 \cdot (\text{alfa} - 1) \cdot (\text{alfa} - 2))
  ElseIf m = 3 Then
    RGMoment = 1 / (beta \cdot 3 \cdot (\text{alfa} - 1) \cdot (\text{alfa} - 2) \cdot (\text{alfa} - 3))
  ElseIf m = 4 Then
    RGMoment = 1 / (beta \cdot 4 \cdot (\text{alfa} - 1) \cdot (\text{alfa} - 2) \cdot (\text{alfa} - 3) \cdot (\text{alfa} - 4))
  Else
    Stop
  End If
Else
  Stop
End If
End Function

Function RGDensity(deriv As Integer, x As Double, _
            alfa As Double, beta As Double) As Double
    'Density function (and derivatives) of reciprocal gamma(alfa, beta)

    Dim f As Double

    f = beta ^(-alfa) * x ^(-alfa - 1) * Exp(-1 / (beta * x)) _
    / Exp(WorksheetFunction.GammaLn(alfa))

    If deriv = 0 Then
        'reciprocal gamma density f(x)
        RGDensity = f
    ElseIf deriv = 1 Then
        '1. order derivative f'(x)
        RGDensity = f * (beta ^(-1) * x ^(-2) - (alfa + 1) * x ^(-1))
    ElseIf deriv = 2 Then
        '2. order derivative f''(x)
        RGDensity = f * (beta ^(-2) * x ^(-4) - 2 * beta ^(-1) * _
                        (alfa + 2) * x ^(-3) + (alfa ^2 + 3 * alfa + 2) * x ^(-2))
    Else
        Stop
    End If

End Function
Appendix D

Vorst

D.1 Visual Basic code

Function Vorst92(sigma As Double, K As Double, n As Integer, _
    q As Double, r As Double, S_0 As Double, T As Double) As Variant
    Dim sigSq As Double, h As Double
    Dim meanGeo As Double, varGeo As Double, EA As Double, EG As Double
    Dim d1 As Double, d2 As Double, priceGeo As Double
    Dim upBound As Double
    Dim Knew As Double, d1new As Double, d2new As Double, price As Double
    Dim output(2) As Double

    'sigma: volatility
    'K:  strike price
    'n:  # averaging times
    'q:  stock dividend
    'r:  interest rate
    'S_0: stock price at time 0
    'T:  time to maturity

    sigSq = sigma ^ 2
    h = T / n  'time increment between averaging dates

    'Mean and variance of logarithm of geometric average
    meanGeo = Log(S_0) + (r - q - sigSq / 2) * (T + h) / 2
    varGeo = sigSq * (h + (T - h) * (2 * n - 1) / (6 * n))

    'Mean of arithmetic and geometric averages
EA = S_0 / n * Exp((r - q) * h) * (1 - Exp((r - q) * n * h)) / (1 - Exp((r - q) * h))
EG = Exp(meanGeo + varGeo / 2)

'Price of geometric Asian option
d1 = (meanGeo - Log(K) + varGeo) / Sqr(varGeo)
d2 = d1 - Sqr(varGeo)
priceGeo = Exp(-r * T) * (EG * WorksheetFunction.NormSDist(d1) - K * WorksheetFunction.NormSDist(d2))

upBound = priceGeo + Exp(-r * T) * (EA - EG)

'Approximate price of Asian option [Vorst, 1992]
Knew = K - (EA - EG) 'Adjusted strike price
d1new = (meanGeo - Log(Knew) + varGeo) / Sqr(varGeo)
d2new = d1new - Sqr(varGeo)
price = Exp(-r * T) * (EG * WorksheetFunction.NormSDist(d1new) - Knew * WorksheetFunction.NormSDist(d2new))

output(0) = price
output(1) = upBound 'upper bound
output(2) = priceGeo 'lower bound

Vorst92 = output

End Function
Appendix E

Curran

E.1 Visual Basic code

Function Curran94(sigma As Double, K As Double, n As Integer, _
    q As Double, r As Double, S_0 As Double, T As Double) As Variant
Dim sigSq As Double, h As Double
Dim meanG As Double, varG As Double
Dim i As Integer
Dim meanS() As Double, varS() As Double, covSG() As Double
Dim d1() As Double, d2 As Double
Dim I1 As Double, I2 As Double, c2 As Double
Dim y As Double, EAG As Double, delta As Double
Dim price As Double
Dim output(1) As Double

'sigma: volatility
'K: strike price
'n: # averaging times
'q: stock dividend
'r: interest rate
'S_0: stock price at time 0
'T: time to maturity

ReDim meanS(n)
ReDim varS(n)
ReDim covSG(n)
ReDim d1(n)
\[ \text{sigSq} = \text{sigma}^2 \]
\[ h = T / n \quad \text{'time increment between averaging dates} \]

'Mean and variance of logarithm of geometric average \( \ln(G) \)
\[ \text{meanG} = \text{Log}(S_0) + (r - q - \text{sigSq} / 2) \times (T + h) / 2 \]
\[ \text{varG} = \text{sigSq} \times h \times (2 \times n + 1) \times (n + 1) / (6 \times n) \]

For \( i = 1 \) To \( n \)

'Mean and variance of logarithm of stock price \( \ln(S) \)
\[ \text{meanS}(i) = \text{Log}(S_0) + (r - q - \text{sigSq} / 2) \times i \times h \]
\[ \text{varS}(i) = \text{sigSq} \times i \times h \]
'Covariance between \( \ln(S) \) and \( \ln(G) \)
\[ \text{covSG}(i) = \text{sigSq} \times h \times ((2 \times n + 1) \times i - i^2) / (2 \times n) \]
Next \( i \)

'Calculate \( L \) to replace \( K \) as the lower limit in the integral
\[ \text{delta} = 10^{-6} \quad \text{'stepsize} \]
\[ y = \text{Log}(K) \quad \text{'initial value (which is too high)} \]
\[ \text{EAG} = \text{CondMean}(y, \text{sigma}, K, n, q, r, S_0, T) \]

Do While EAG > K
\[ y = y - \text{delta} \]
\[ \text{EAG} = \text{CondMean}(y, \text{sigma}, K, n, q, r, S_0, T) \]
Loop

'At this point \( \text{EAG} < K \) so we use the last but one \( y \)
\[ y = y + \text{delta} \]
\[ L = \text{Exp}(y) \]

'Calculate option price
\[ I1 = 0 \]
For \( i = 1 \) To \( n \)
\[ d1(i) = (\text{meanG} - \text{Log}(L) + \text{covSG}(i)) / \text{Sqr}(\text{varG}) \]
\[ I1 = I1 + \text{Exp}(\text{meanS}(i) + \text{varS}(i) / 2) \times \]
\[ * \text{WorksheetFunction.NormSDist}(d1(i)) \]
Next \( i \)
\[ I1 = I1 / n \]
\[ d2 = (\text{meanG} - \text{Log}(L)) / \text{Sqr}(\text{varG}) \]
\[ I2 = K \times \text{WorksheetFunction.NormSDist}(d2) \]
E.1. VISUAL BASIC CODE

```vbnet
price = Exp(-r * T) * (I1 - I2)

output(0) = price
output(1) = L

Curran94 = output()

End Function

Function CondMean(y As Double, sigma As Double, K As Double, n As Integer, _
    q As Double, r As Double, S_0 As Double, T As Double) As Double
    Dim sigSq As Double, h As Double
    Dim meanG As Double, varG As Double
    Dim i As Integer
    Dim meanS() As Double, varS() As Double, covSG() As Double
    Dim result As Double

    ReDim meanS(n)
    ReDim varS(n)
    ReDim covSG(n)

    sigSq = sigma ^ 2
    h = T / n 'time increment between averaging dates

    'Mean and variance of logarithm of geometric average ln(G)
    meanG = Log(S_0) + (r - q - sigSq / 2) * (T + h) / 2
    varG = sigSq * h * (2 * n + 1) * (n + 1) / (6 * n)

    For i = 1 To n
        'Mean and variance of logarithm of stock price ln(S)
        meanS(i) = Log(S_0) + (r - q - sigSq / 2) * i * h
        varS(i) = sigSq * i * h
        'Covariance between ln(S) and ln(G)
        covSG(i) = sigSq * h * ((2 * n + 1) * i - i ^ 2) / (2 * n)
    Next i

    result = 0
    For i = 1 To n
        result = result + Exp(meanS(i) + (y - meanG) * covSG(i) / varG _
            + (varS(i) - covSG(i) ^ 2 / varG) / 2)
    Next i
```
result = result / n

CondMean = result

End Function
Bibliography


