Exploring Volatility Derivatives: New Advances in Modelling

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1. Volatility Products
Historical Volatility Products

- Historical variance: \( \frac{1}{n} \sum_{i=1}^{n} \ln\left(\frac{S_i}{S_{i-1}}\right)^2 \)

- OTC products:
  - Volatility swap
  - Variance swap
  - Corridor variance swap
  - Options on volatility/variance
  - Volatility swap again

- Listed Products:
  - Futures on realized variance
Implied Volatility Products

• Definition
  – Implied volatility: input in Black-Scholes formula to recover market price:
  – Old VIX: proxy for ATM implied vol
  – New VIX: proxy for variance swap rate

• OTC products
  – Swaps and options

• Listed products
  – VIX Futures contract
  – Volax
1. Volatility Products: \( VIX \) Futures Pricing
Vanilla Options

Simple product, but complex mix of underlying and volatility:

Call option has:
- Sensitivity to $S$: $\Delta$
- Sensitivity to $\sigma$: Vega

These sensitivities vary through time and spot, and vol:
To play pure volatility games (eg bet that S&P vol goes up, no view on the S&P itself)

- Need of constant sensitivity to vol
- Achieved by combining several strikes;
- Ideally achieved by a log profile: (variance swaps)
• Under BS: \( dS = \sigma S dW \), \( \mathbb{E} [ \ln \frac{S_T}{S_0} ] = -\frac{\sigma^2}{2} T \)

• For all \( S \),

\[
\ln \frac{S}{S_0} = \frac{S - S_0}{S_0} - \int_0^{S_0} \frac{(K - S)^+}{K^2} dK - \int_{S_0}^{\infty} \frac{(S - K)^+}{K^2} dK
\]

• The log profile is decomposed as:

\[
\frac{1}{S_0} \text{Futures} - \int_0^{S_0} \frac{P_{K,T}}{K^2} dK - \int_{S_0}^{\infty} \frac{C_{K,T}}{K^2} dK
\]

• In practice, finite number of strikes \( \Rightarrow \) CBOE definition:

\[
VIX_t^2 \equiv \frac{2}{T} \sum \frac{K_{i+1} - K_{i-1}}{2K_i^2} e^{rT} X(K_i, T) - \frac{1}{T} \left( \frac{F}{K_0} - 1 \right)^2
\]

where \( X \) is a Put if \( K_i < F \), a Call otherwise
### Option prices for one maturity

**SPX** ↓1095.45 +8.33  
At 16:59  Op 1087.12 Hi 1095.69 Lo 1087.12

<table>
<thead>
<tr>
<th>Ticker</th>
<th>Strike</th>
<th>Bid</th>
<th>Ask</th>
<th>Last</th>
<th>Volume</th>
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<td>1040</td>
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<td>1239</td>
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<td>SPT-EE</td>
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</table>

**Index OMEN**

**Option Monitor: S&P 500 INDEX**

Center **1095.45** Number of Strikes **16** or % from Center **Exchange C** (Composite)
Perfect Replication of $VIX^2_{T_1}$

\[ VIX^2_{T_1} = -\frac{2}{\delta T} \text{price}_t \left( \ln \frac{S_{T_1+\delta T}}{S_{T_1}} \right) \]
\[ = \text{price}_t \left[ \frac{2}{\delta T} \ln \frac{S_{T_1}}{S_{T_0}} - 2 \ln \frac{S_{T_1+\delta T}}{S_{T_0}} \right] \]
\[ = \text{price}_t[PF] \]

We can buy today a PF which gives $VIX^2_{T_1}$ at $T_1$: buy $T_2$ options and sell $T_1$ options.
Theoretical Pricing of $VIX$ Futures $F^{VIX}$ before launch

- $F_t^{VIX}$: price at $t$ of receiving $\sqrt{PF_t} = VIX_t = F_{T1}^{VIX}$ at $T_1$

$$F_t^{VIX} = \mathbb{E}[\sqrt{PF_t}] \leq \sqrt{\mathbb{E}_t[PF_t]} = \sqrt{PF_t} = \text{Upper Bound (UB)}$$

- The difference between both sides depends on the variance of PF (vol vol), which is difficult to estimate.
Pricing of $F^{VIX}$ after launch

Much less transaction costs on F than on PF (by a factor of at least 20)

$\rightarrow$ Replicate PF by F instead of F by PF!

$$PF_{T_1} = (F_{T_1}^{VIX})^2 = (F_t^{VIX})^2 + 2 \int_t^{T_1} (F_s^{VIX} - F_t^{VIX}) dF_s^{VIX} + QV_{t,T_1}$$

$$PF_t = \mathbb{E}_t[(F_{t_1}^{VIX})^2] = \mathbb{E}_t[F_{T_1}^{VIX}]^2 + \text{Var}_t[F_{T_1}^{VIX}]$$

$$\implies F_t^{VIX} = \mathbb{E}_t[F_{T_1}^{VIX}] = \sqrt{PF_t - \text{Var}_t[F_{T_1}^{VIX}]}(\leq \sqrt{PF_t} = UB)$$
Bias estimation

\[ F_t^{VIX} = \sqrt{UB^2 - Var_t[F_{T_1}^{VIX}]} \]

- \(Var[F_{T_1}]\) can be estimated by combining the historical volatilities of F and Spot VIX.
- Seemingly circular analysis: F is estimated through its own volatility!

Example: 192 = \(\sqrt{200^2 - 56^2}\)
### VIX Futures Fair Value

<table>
<thead>
<tr>
<th>Ticker</th>
<th>Expiration</th>
<th>Days</th>
<th>Risk Free</th>
<th>Upper Bound</th>
<th>Volatility of Vix</th>
<th>Fair Value</th>
<th>Futures Price</th>
<th>Fair - Futures</th>
</tr>
</thead>
<tbody>
<tr>
<td>UX4</td>
<td>11/17/04</td>
<td>12</td>
<td>1.65%</td>
<td>141.99</td>
<td>82.42</td>
<td>140.44</td>
<td>138.20</td>
<td>2.24</td>
</tr>
<tr>
<td></td>
<td>12/18/04*</td>
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<td>1.74%</td>
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<td>Historical</td>
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<td></td>
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<tr>
<td>UX5</td>
<td>1/19/05</td>
<td>75</td>
<td>1.88%</td>
<td>164.05</td>
<td>66.44</td>
<td>156.80</td>
<td>152.00</td>
<td>4.80</td>
</tr>
<tr>
<td></td>
<td>3/19/05*</td>
<td></td>
<td>2.04%</td>
<td></td>
<td>Historical</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UXG5</td>
<td>2/15/05</td>
<td>103</td>
<td>1.97%</td>
<td>164.05</td>
<td>64.45</td>
<td>154.73</td>
<td>157.30</td>
<td>-2.57</td>
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<td>3/19/05*</td>
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<td>2.04%</td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>UX5</td>
<td>5/18/05</td>
<td>194</td>
<td>2.15%</td>
<td>163.50</td>
<td>58.87</td>
<td>149.13</td>
<td>170.00</td>
<td>-20.87</td>
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<tr>
<td></td>
<td>6/18/05*</td>
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<td>2.19%</td>
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<td>Historical</td>
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</table>

* Options of these additional maturities are used to compute the Fair Price of the corresponding futures contracts.

<table>
<thead>
<tr>
<th>VXB Fast</th>
<th>VXB Spot</th>
<th>Fast - Spot</th>
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</thead>
<tbody>
<tr>
<td>140.06</td>
<td>140.30</td>
<td>-0.24</td>
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**Behind The Scene**

<table>
<thead>
<tr>
<th>Futures Ticker</th>
<th>Expiry</th>
<th>Bid</th>
<th>Mid</th>
<th>Ask</th>
<th>Volvol</th>
<th>Fair Value</th>
<th>Bid</th>
<th>Futures Last Price</th>
<th>Ask</th>
<th>FAIR-PRICE</th>
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<tbody>
<tr>
<td>UX1 Index</td>
<td>11/17/2004</td>
<td>123.96</td>
<td>142.33</td>
<td>150.59</td>
<td>60.00%</td>
<td>141.49</td>
<td>136.8</td>
<td>138.2</td>
<td>138.5</td>
<td>3.29</td>
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<td>UX2 Index</td>
<td>1/19/2005</td>
<td>138.60</td>
<td>164.30</td>
<td>186.49</td>
<td>50.00%</td>
<td>160.14</td>
<td>151</td>
<td>152</td>
<td>152</td>
<td>8.14</td>
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<tr>
<td>UX3 Index</td>
<td>2/16/2005</td>
<td>138.60</td>
<td>164.30</td>
<td>186.49</td>
<td>40.00%</td>
<td>160.63</td>
<td>157</td>
<td>158.3</td>
<td>158.2</td>
<td>2.33</td>
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<td>UX4 Index</td>
<td>5/18/2005</td>
<td>139.50</td>
<td>163.38</td>
<td>184.20</td>
<td>30.00%</td>
<td>159.52</td>
<td>169</td>
<td>170</td>
<td>170</td>
<td>-10.48</td>
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</table>

**Synthetic Variance swap rates**

<table>
<thead>
<tr>
<th>Expiration</th>
<th>Bid</th>
<th>Mid</th>
<th>Ask</th>
</tr>
</thead>
<tbody>
<tr>
<td>12/18/2004</td>
<td>13.17</td>
<td>14.08</td>
<td>14.93</td>
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<tr>
<td>1/22/2005</td>
<td>13.51</td>
<td>14.23</td>
<td>14.91</td>
</tr>
<tr>
<td>9/17/2005</td>
<td>15.82</td>
<td>16.28</td>
<td>16.72</td>
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</table>

**Realized Variance Futures (3-month)**

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<thead>
<tr>
<th>Expiration</th>
<th>Mkt Quote</th>
<th>Mid at bid</th>
</tr>
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<tbody>
<tr>
<td>12/18/2004</td>
<td>161.00</td>
<td>166.00</td>
</tr>
<tr>
<td>3/19/2005</td>
<td>231.00</td>
<td>238.50</td>
</tr>
<tr>
<td>6/18/2005</td>
<td>266.50</td>
<td>271.00</td>
</tr>
<tr>
<td>9/18/2004</td>
<td>289.50</td>
<td>293.50</td>
</tr>
<tr>
<td>9/18/2004</td>
<td>166.79</td>
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</tr>
</tbody>
</table>

**VIX Futures Fair Value**

<table>
<thead>
<tr>
<th>VXB Fast</th>
<th>VXB spot</th>
<th>SPOT-FAST</th>
</tr>
</thead>
<tbody>
<tr>
<td>140.217</td>
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</tbody>
</table>

**Variance swap term structure**

**Forward Variance term structure**

**Implied Vol skew**

**Forward Variance swap rates**

<table>
<thead>
<tr>
<th>Expiration</th>
<th>Bid</th>
<th>Mid</th>
<th>Ask</th>
</tr>
</thead>
<tbody>
<tr>
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<td>12.39</td>
<td>14.23</td>
<td>15.86</td>
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<tr>
<td>12/18/2004</td>
<td>11.59</td>
<td>14.40</td>
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<td>1/22/2005</td>
<td>11.59</td>
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<td>1/22/2005</td>
<td>13.86</td>
<td>16.43</td>
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<td>13.95</td>
<td>16.34</td>
<td>18.42</td>
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<td>6/18/2005</td>
<td>13.95</td>
<td>16.34</td>
<td>18.42</td>
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<td>6/18/2005</td>
<td>14.85</td>
<td>17.68</td>
<td>20.11</td>
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<tr>
<td>9/17/2005</td>
<td>14.85</td>
<td>17.68</td>
<td>20.11</td>
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VIX Summary

• VIX Futures is a FWD volatility between future dates $T_1$ and $T_2$.

• Depends on volatilities over $T_1$ and $T_2$.

• Can be locked in by trading options maturities $T_1$ and $T_2$.

• 2 problems:
  – Need to use all strikes (log profile)
  – Locks in $\sigma^2$, not $\sigma$ → need for convexity adjustment and dynamic hedging.
2. Linking Various Volatility Products
Volatility as an Asset Class: A Rich Playfield

- Options on $S$ ($C(S)$)
- OTC Variance/Vol Swaps ($\text{VarS/VolS}$)
  - Square of historical vol up to maturity
- Futures on Realised Variance ($\text{RV}$)
  - Square of historical vol over a future quarter
- Futures on Implied ($\text{VIX}$)
- Options on Variance/Vol Swaps ($C(\text{Var S})$)
• The pay-off of an OTC Variance Swap can be replicated by a string of Realized Variance Futures:

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
\text{t} & \text{T}_0 & \text{T}_1 & \text{T}_2 & \text{T}_3 & \text{T}_4 & \text{T} \\
& & & & & \\
& & & & & \\
\end{array}
\]

• From 12/02/04 to maturity 09/17/05, bid-ask in vol: 15.03/15.33

• Spread=.30% in vol, much tighter than the typical 1% from the OTC market
• Assume that RV and VIX, with prices RV and F are defined on the same future period $[T_1, T_2]$

• If at $T_0$, $RV_0 < F_0^2$ then buy 1 RV Futures and sell 2 $F_0$ VIX Futures

• At $T_1$,

$$PL_1 = RV_1 - RV_0 - 2F_0(F_1 - F_0)$$

$$> RV_1 - F_0^2 - 2F_0(F_1 - F_0)$$

$$= RV_1 - F_1^2 + (F_1 - F_0)^2$$

• If $RV_1 < F_1^2$ sell the PF of options for $F_1^2$ and Delta hedge in S until maturity to replicate RV.

• In practice, maturities differ: conduct the same approach with a string of VIX Futures
3. Volatility Modeling
Volatility Modeling

- Neuberger (90): Quadratic variation can be replicated by delta hedging Log profiles
- Dupire (92): Forward variance synthesized from European options. Risk neutral dynamics of volatility to fit the implied vol term structure. Arbitrage pricing of claims on Spot and on vol
- Heston (93): Parametric stochastic volatility model with quasi closed form solution
- Dupire (96), Derman-Kani (97): non parametric stochastic volatility model with perfect fit to the market (HJM approach)
• Matytsin (99): Parametric stochastic volatility model with jumps to price vol derivatives

• Carr-Lee (03), Friz-Gatheral (04): price and hedge of vol derivatives under assumption of uncorrelated spot and vol increments

• Duanmu (04): price and hedge of vol derivatives under assumption of volatility of variance swap

• Dupire (04): Universal arbitrage bounds for vol derivatives under the sole assumption of continuity
Variance swap based approach (Dupire (92), Duanmu (04))

• \( V = QV(0,T) \) is replicable with a delta hedged log profile (parabola profile for absolute quadratic variation)
  
  – Delta hedge removes first order risk
  
  – Second order risk is unhedged. It gives the quadratic variation

• \( V \) is tradable and is the underlying of the vol derivative, which can be hedged with a position in \( V \)

• Hedge in \( V \) is dynamic and requires assumptions on

\[
V_t = \mathbb{E}[V] = QV_{0,t} + \mathbb{E}_t[QV_{t,T}]
\]
Stochastic Volatility Models

• Typically model the volatility of volatility (volvol). Popular example: Heston (93)

\[
\frac{dS_t}{S_t} = \sqrt{\nu_t}dW_t
\]

\[
d\nu_t = \kappa(\nu_\infty - \nu_t)dt + \alpha\sqrt{\nu_t}dZ_t
\]

• Theoretically: gives unique price of vol derivatives (1st equation can be discarded), but does not provide a natural unique hedge

• Problem: even for a market calibrated model, disconnection between volvol and real cost of hedge.
• A pronounced skew imposes a high spot/vol correlation and hence a high volvol if the vol is high

• As will be seen later, non flat smiles impose a lower bound on the variability of the quadratic variation

• High spot/vol correlation means that options on $S$ are related to options on vol: do not discard 1st equation anymore

From now on, we assume 0 interest rates, no dividends and $V$ is the quadratic variation of the price process (not of its log anymore)
Carr-Lee approach

- Assumes
  - Continuous price
  - Uncorrelated increments of spot and of vol
- Conditionally to a path of vol, $X(T)$ is normally distributed, $= X_0 + \sqrt{V} g$ (g: normal sample)
- Then it is possible to recover from the risk neutral density of $X(T)$ the risk neutral density of V
- Example: $\mathbb{E}[(X_T - X_0)^{2n}] = \mathbb{E}[V^n g^{2n}] = \mu_{2n} \mathbb{E}[V^n]$
4. Lower Bound
Densities of $X$ and $V$

- How can we link the densities of the spot and of the quadratic variation $V$? What information do the prices of vanillas give us on the price of vol derivatives?
- Variance swap based approach: no direct link
- Stochastic vol approach: the calibration to the market gives parameters that determines the dynamics of $V$
- Carr-Lee approach: uncorrelated increments of spot and vol gives perfect reading of density of $X$ from density of $V$
• Claims can be on the forward quadratic variation
• Extreme case: $f(\nu_T)$ where $\nu_T$ is the instantaneous variance at $T$
• If $f$ is convex,

$$E[f(\nu_T)] = E[E[f(\nu_T|X_T = K)]] \geq E[f(E[\nu_T|X_T = K])] = E[f(\nu_{loc}(K, T))]$$

Which is a quantity observable from current option prices
\( X(T) \) not normal \( \Rightarrow \) \( V \) not constant

- Main point: departure from normality for \( X(T) \) enforces departure from constancy for \( V \), or smile non flat \( \Rightarrow \) variability of \( V \)

- Carr-Lee: conditionally to a path of vol, \( X(T) \) is gaussian

- Actually, in general, if \( X \) is a continuous local martingale
  - \( QV(T) = \text{constant} \Rightarrow X(T) \) is gaussian
  - \textbf{Not}: conditional to \( QV(T) = \text{constant} \), \( X(T) \) is gaussian
  - \textbf{Not}: \( X(T) \) is gaussian \( \Rightarrow QV(T) = \text{constant} \)
The Main Argument

• If you sell a convex claim on $X$ and delta hedge it, the risk is mostly on excessive realized quadratic variation

• Hedge: buy a Call on $V$!

• Classical delta hedge (at a constant implied vol) gives a final P&L that depends on the Gammas encountered

• Perform instead a “business time” delta hedge: the payoff is replicated as long as the quadratic variation is not exhausted
• Extend $f(x)$ to $f(x, \nu)$ as the Bachelier (normal BS) price of $f$ for start price $x$ and variance $\nu$:

$$
f(x, \nu) \equiv \mathbb{E}^{x,\nu}[f(X)] \equiv \frac{1}{\sqrt{(2\pi\nu)}} \int f(y) e^{-\frac{(y-x)^2}{2\nu}} dy
$$

with $f(x, 0) = f(x)$

• Then, $f'_\nu(x, \nu) = \frac{1}{2} f_{xx}(x, \nu)$

• We explore various delta hedging strategies
Calendar Time Delta Hedging

- Delta hedging with constant vol: P&L depends on the path of the volatility and on the path of the spot price.

\[
df(X_t, \sigma(T - t)) = f_x dX_t - \sigma^2 f_\nu dt + \frac{1}{2} f_{xx} dQV_{0,t}
\]

\[
= f_x dX_t + \frac{1}{2} f_{xx} (dQV_{0,t} - \sigma^2 dt)
\]

- Calendar time delta hedge: replication cost of

\[
f(X_0, \sigma^2 T) + \frac{1}{2} \int_0^T f_{xx}(dQV_{0,u} - \sigma^2 du)
\]

- In particular, for \(\sigma = 0\), replication cost of \(f(X_t)\)

\[
f(X_0) + \frac{1}{2} \int_0^T f_{xx} dQV_{0,u}
\]
Delta hedging according to the quadratic variation: P&L that depends only on quadratic variation and spot price

\[ df(X_t, L - QV_{0,t}) = f_x dX_t - f_{\nu} dQV_{0,t} + \frac{1}{2} f_{xx} dQV_{0,t} = f_x dX_t \]

Hence, for \( QV_{0,T} \leq L \)

\[ f(X_t, L - QV_{0,t}) = f(X_0, L) + \int_0^t f_x(X_u, L - QV_{0,u}) dX_t \]

And the replicating cost of \( f(X_t, L - QV_{0,t}) \) is \( f(X_0, L) \)

\( f(X_0, L) \) finances exactly the replication of \( f \) until \( \tau : QV_{0,\tau} = L \)
Hedge with Variance Call

- Start from $f(X_0, L)$ and delta hedge $f$ in “business time”

- If $V < L$, you have been able to conduct the replication until $T$ and your wealth is $f(X_T, L - V) \geq f(X_T)$

- If $V > L$, you “run out of quadratic variation” at $\tau < T$. If you then replicate $f$ with 0 vol until $T$, extra cost:

  $$\frac{1}{2} \int_{\tau}^{T} f''(X_T) dQV_t \leq \frac{M_f}{2} \int_{\tau}^{T} dQV_t = \frac{M_f}{2}(V - L)$$

  where $M_f \equiv \sup f''(x)$

- After appropriate delta hedge, $f(X_0, L) + \frac{M}{2} CallL^V$ dominates $f(X_T)$ which has a market price $f(X_0, L\hat{f})$
Lower Bound for Variance Call

- \( C^V_L \): price of a variance call of strike \( L \). For all \( f \),

\[
C^V_L \geq \frac{2}{M_f} (f(X_0, L^f) - f(X_0, L))
\]

- We maximize the RHS for, say, \( M_f \leq 2 \)

- We decompose \( f \) as

\[
f(x) = f(X_0) + (x - X_0)f'(X_0) + \int f''(K)V_{\text{vanilla}}_K(x) dK
\]

Where \( V_{\text{vanilla}}_K(x) \equiv K - x \) if \( K \leq X_0 \) and \( x - K \) otherwise.

Then, \( C^V_L \geq \int f''(K)(V_{\text{vanilla}}_K(L^K) - V_{\text{vanilla}}_K(L)) dK \)

where \( C^V_L \) is the price of \( V_{\text{vanilla}}_K(x) \) for variance \( V \) and \( L^K \) is the market implied variance for strike \( K \)
Lower Bound Strategy

- Maximum when \( f'' = 2 \) on \( A \equiv K : L^K \geq L \), 0 elsewhere
- then, \( f(x) = 2 \int_A V\text{vanilla}_K(x) dK \) (truncated parabola) and \( C_L^V \geq 2 \int_A (V\text{an}_K(L^K) - V\text{an}_K(L)) dK \)
Arbitrage Summary

- If a Variance Call of strike \( L \) and maturity \( T \) is below its lower bound:
  - 1) at \( t=0 \),
    - Buy the variance call
    - Sell all options with implied vol \( \leq \sqrt{\frac{L}{T}} \)
  - 2) between 0 and \( T \),
    - Delta hedge the options in business time
    - If \( \tau < T \), then carry on the hedge with 0 vol
  - 3) at \( T \), sure again
5. Conclusion
• Skew denotes a correlation between price and vol, which links options on prices and on vol

• Business time delta hedge links P&L to quadratic variation

• We obtain a lower bound which can be seen as the real intrinsic value of the option

• Uncertainty on $V$ comes from a spot correlated component (IV) and an uncorrelated one (TV)

• It is important to use a model calibrated to the whole smile, to get IV right and to hedge it properly to lock it in